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AN
EASY INTRODUCTION
TO
ALGEBRA,

WITH NOTES,

WHEREIN THE RULES ARE DEMONSTRATED,
AND THE OPERATIONS EXPLAINED,
ADAPTED TO THE USE OF SCHOOLS,

AND

Those who study without a Tutor.

TO WHICH IS PREFIXED,

An Essay on the Uses of the Mathematics,
WITH DIRECTIONS,

To assist the Learner in their Attainment.

By CHARLES BUTLER,

TEACHER OF THE MATHEMATICS AT CREAM SCHOOL.

“ALGEBRA confers the power of invention and combination beyond any other study. The power of readily calling up possibilities before the imagination, of contrasting them with realities, and with one another, and of deciding on their respective merits, appears to me the highest state of perfection, at which our faculties can arrive.—Now an acute observation, and full conception of things, joined to a habit of comparing and combining them, not unlike that casting about of the thoughts which takes place in the solution of Algebraic problems, is the only way, I believe, in which this inestimable talent can be acquired.

Dr. BRIDGES.

London:

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1799.



TO
RICHARD BERENS, Esq.

SIR,

EVERY species of knowledge is good and valuable in proportion as it is conducive to public benefit. That the mathematical sciences are eminently so, is a truth with which you are well acquainted: this added to the clearness of the reasoning employed, and the certainty of the conclusions obtained, were motives sufficient to induce you to prosecute these studies with an ardour suited to their importance.

To cultivate those excellent qualities, which ultimately promote the good of society, is the characteristic of a noble and generous mind. This has distinguished
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those of your worthy relatives, whom I have the honour of knowing; and it is but justice to acknowledge, that you have done credit to their example.

THAT you may ever enjoy that pleasure which results from rectitude of heart and an improved understanding, is the sincere and hearty wish of,

SIR,

Your most obliged

Humble Servant,

CHARLES BUTLER.

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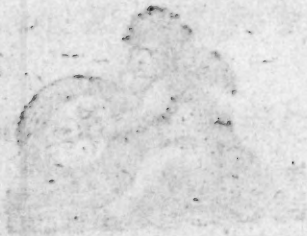
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PREFACE.

SECRETARY

of the
General Assembly
of the State of New York



PREFACE

PREFACE.

THE spirit of inquiry which so eminently distinguishes the present age, has rendered the mathematics a necessary part of learning; for in proportion as natural philosophy and the useful arts have advanced, the study of these sciences has become more generally requisite; and the means of communicating them having from time to time been varied, extended, and improved, this branch of education might at first-sight be supposed complete.

But an attentive review of the elementary books on Algebra, and a little experience in teaching, will soon convince us that every thing necessary to the purposes of instruction has not yet been accomplished. The writings of Newton, Saun-

derfon, Maclaurin, Emerfon and Simpson, have been, and ftill are juftly confidered as patterns of elegance and perfeftion, and as fuch, are neceffary to complete the ftudies of thofe who would become great proficient; but to minds accuftomed to abfttract reasoning, and constantly bufied in fpeculations above the reach of ordinary capacities, it muft be a very difficult task to ftoop to the wants of the novice, accordingly we find in their valuable books very little of the kind of information wanted by a meer beginner. Thofe authors, it is true, have proceeded regularly from firft principles, but they have treated of the elementary parts with fuch exceffive brevity, as leads us to imagine, that thefe were admitted rather for the fake of method, than as adequate introductions to the abftufe and difficult matters which follow.

To remedy this defect, is a task which has fallen to the lot of others, who perhaps, were not fo well qualified to excel in the higher departments of literature; and although it muft be acknowledged, that introductory treatifes of great merit have appeared, yet it happens, that in the greater number, either too little, or too much has
been

P R E F A C E.

iii

been done, by which means the tutor is subjected to incessant drudgery ; or the whole work being performed, the pupil has little else left to exercise his abilities than merely to transcribe.

The present work is not to be considered as a complete treatise, or as entering very deeply into the subject (although what is here given will be sufficient for most learners,) but rather as a plain and easy introduction, suited to remove some of those obstacles which usually retard the progress of the scholar at his first setting out, and to furnish him with information sufficient to render an application to the tutor seldom necessary ; in this respect, it will be found useful to those who are without the advantages of verbal instruction. Here every thing is made plain and easy, phrases and technical terms are as much as possible avoided, the rules and explanations are written in familiar language, many of the examples are wrought out at length and explained, some are wrought out without explanation, and others are left entirely to the skill and sagacity of the learner.

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But however the present work may suit the capacity of a learner, it will be necessary to bespeak the candour and indulgence of the man of science: he will probably expect to see the several parts exactly fitted and proportioned like those of a well finished machine, so as to constitute one complete and entire whole; in this I fear he will be disappointed: a strict scientific arrangement, however desirable, can seldom be admitted without prejudice into books of rudiments. In these whatever is difficult must in the beginning be kept out of sight, and the matters disposed not in the exact order in which the mathematician contemplates them, but in that which is best suited to the comprehension of the learner. In these the beauty and elegance of the whole must yield to a minute and circumstantial attention to the parts, and a scientific brevity of stile give place to circumlocution, and even sometimes to tautology.

The subjects are accordingly here arranged, not in the usual order, but that in which it is judged they will be best understood; and concise and elegant demonstrations have sometimes been omitted for others less so, when it was supposed the latter would be easier comprehended.

My

P R E F A C E.

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My best acknowledgments are due to all who have in any manner contributed to the present design, particularly to the Rev. Mr. Gilpin, and many gentlemen who have been, or still are members of Cheam school. How far the book will answer the intention with which it was written, experience alone will prove, and whether the author is entitled to indulgence or censure must be left for others to determine. Of the typographical errors, the usual plea, of distance from the press, may with great truth be alledged as the cause; and for other imperfections, excuses might be offered: but as the asperity of critical censure is seldom softened by apologies, the work is left to plead its own cause.

Vous trouverez dans cette ouvrage-ci
Du passable, du bon, et du mauvais aussi:
C'est sur ce pied qu'on vous le livre,
Lecteur, attendez-vous-y bien,
Voilà le portrait de tout livre,
Comme c'est le portrait du mien.

DU CERCEAU.

On the Use of Mathematical Learning, with directions to assist those who study without a tutor.

To teach the young idea how to shoot,
To pour the fresh instruction o'er the mind,
——— to fix
The generous purpose in the glowing breast.

THOMSON.

THE excellency and usefulness of mathematical learning, having been largely insisted on by writers of the first abilities, little new can be expected on the subject; but to those whose reading has not extended to the works alluded to, a repetition of the sentiments contained in them will not be unacceptable, and will be fully sufficient for the present purpose, being calculated to excite in the attentive mind a thirst for useful knowledge.

Mathematics is a science, which treats of the properties of magnitude, so that whatever is extended, or (in other words) whatever occupies space, is the subject of mathematical reasoning: now every thing we see or touch, or of which our senses can take any cognizance, fills space; there-

therefore mathematical inquiry extends to every thing in nature.

Geometry is the first and principal branch of the mathematics; it treats of extension or magnitude, under the form of lines, superficies, and solids. Subordinate to Geometry are Algebra and Arithmetic, these may be considered as one and the same, since both treat of the calculations of quantities; the first by general characters, and the latter by numbers.

These therefore, which treat of magnitude and multitude, are the foundation of all the other branches, and are called pure or abstract mathematics, because they investigate and demonstrate the properties of magnitudes and quantities, considered purely as such, without regard to their application, supposing them distinct, and apart from all substance whatever. But when pure mathematics is applied to particular subjects or uses in life, as in computing the length of a rope or way, the dimensions of a field or house, the height of a tree or a tower, these constitute mixed or practical mathematics, a very extensive branch,

branch, comprehending many others, as astronomy, geography, navigation, trigonometry, mensuration, optics, perspective, hydrostatics, surveying, architecture, fortification, gunnery, &c. &c. &c. with all that part of natural philosophy which treats of motions, velocities, forces and effects in general*.

Hence many of the necessities of life, as well as its conveniences and luxuries, are obtained through the medium of mathematical knowledge. To this the architect owes the contrivance and execution of his plans, and every building, from the sumptuous palace to the peasant's cot, owes its existence. By this, stately ships, which protect our coasts or waft to our shores the produce of foreign climes, are constructed; by it the hardy sailor shapes his course across the pathless ocean, and the intrepid foldier contrives his operations for the annoyance of our enemies, or to defend us against them.—The sable miner ransacks the bowels of the earth to procure metals for our use, or fuel for our fires; and the ingenious mechanic adjusts the various

* *Mathesin philosophiam naturalem terminare debere.*

machines contrived for public benefit, and the abridgement of manual labour. To mathematical principles almost every instrument we use is indebted for its perfections, and by them the extent and value of property of every kind is accurately ascertained.

Hence it will appear that mathematics, considered with respect to its immediate application, is indispensibly necessary to complete the philosopher, the mariner, the soldier, and the artist; but it is by no means confined to practical arts, and necessary computations*; it has another object of equal if not greater importance, which renders it worthy the attention of all—the mathematics is eminently serviceable to strengthen, and improve the intellectual faculties, and fit them for every kind of speculation. “Would you, says Mr. Locket, have a man reason well, you must use him to it betimes, exercise his mind in observing the connection of ideas, and following them in train; nothing does this better than mathematics, which, therefore, I think should be taught all

* Jones on Mathematical Learning.

† Conduct of the Understanding.

those who have time and opportunity, not so much to make them mathematicians, as to make them reasonable creatures."

* It is the habit of reasoning that makes a reasoner, and therefore the true way to acquire this talent, is by being much conversant in those sciences where the art of reasoning is allowed to reign in the greatest perfection; hence it was that the ancients, who so well understood the manner of forming the mind, always began with the mathematics, as the foundation of their philosophical studies: Here the understanding is by degrees habituated to truth, contracts insensibly a certain fondness for it, and learns never to yield its assent to any proposition but where the evidence is sufficient to produce full conviction. For this reason Plato has called mathematical demonstrations *The Cathartics or Purgatives of the Soul*, being the proper means to cleanse it from error, and restore that natural exercise of its faculties†.

* Duncan.

† Notum est Platonem in gymnasii sui limine---Nemo huc Pedem inferat nisi Geometres, inscripisse. *Rehaulti Physica.*

If,

If, therefore, we would form our minds to a habit of reasoning closely and in train, we cannot take any more certain method, than the exercising ourselves in mathematical demonstrations, so as to contract a kind of familiarity with them; not that we look upon it as necessary that all men should be deep mathematicians, but that having got the way of reasoning which that study necessarily brings the mind to, they may be able to transfer it to other parts of knowledge as they shall have occasion; for in all sorts of reasoning, every single argument should be managed as a mathematical demonstration*.

I shall sum up the whole in the expressive words of the excellent Dr. Barrow†. “The mathematics, says he, effectually exercise, not vainly delude, nor vexatiously torment studious minds with obscure subtilties, but plainly demonstrate every thing within their reach, draw certain conclusions, instruct by profitable rules, and unfold pleasant questions. These disciplines likewise enure and

* Locke on the Understanding.

† See his inaugural oration on being appointed Professor of Mathematics at Cambridge, 1663.

corroborate the mind to a constant diligence in study; they wholly deliver us from a credulous simplicity, most strongly fortify us against the vanity of scepticism, effectually restrain us from a rash presumption, most easily incline us to a due assent, and perfectly subject us to the government of right reason. While the mind is abstracted and elevated from sensible matter, distinctly views pure forms, conceives the beauty of ideas, and investigates the harmony of proportions, the manners themselves are insensibly corrected and improved, the affections composed and rectified, the fancy calmed and settled, and the understanding raised and exalted to more divine contemplations."

Of the first Principles, with an explication of the terms made use of.

IT is the business of science, says an ingenious writer, from a few general principles to draw a great number of particular conclusions—these being regularly deduced are employed as principles and other conclusions drawn from them, again from these others are obtained, and so on without end—it will be necessary for us to

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shew what those principles are, how the conclusions are derived, and to explain the terms made use of.

There are in the Mathematics, definitions, postulates, axioms, propositions, problems, theorems, lemmas, corollaries, and scholiums. The first principles of science are definitions, from these by means of legitimate reasoning (viz. by deductions fairly made) the truths which constitute science are derived.

A definition is the explication of any term or word, shewing the sense in which it is to be understood, or it is an enumeration of those properties which constitute the thing defined, and distinguish it from every thing else.

* A proposition is that in which something is either to be done, or to be proved, in the former case it is called a problem and in the latter a theorem.

When

* The word proposition in its general acceptance is restrained to what Euclid calls a *theorem*, in every *theorem* there is what logicians call the *subject* and the *predicate*. So prop. 37. book 1. *Triangles on the same base and between the same parallels* is the *subject*, and it is *predicated* of such triangles that *they are equal*.

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A converse

When what the problem requires to be done is so manifestly possible as to want no argument to prove it, such a problem is named a postulate.

In like manner when the truth affirmed in a theorem is so self evident as to require no proof, such a theorem is named an axiom.

A demonstration is that process by which the truth affirmed in the proposition is clearly made out and proved.

A converse proposition is one whose *subject* is the *predicate* of a former proposition, and whose *predicate* is the *subject* of that proposition. Thus prop. 5. may be expressed in this manner, *if two sides of a triangle are equal, the angles opposite them will be equal*, and prop. 6. is its converse, viz. *If two angles of a triangle are equal, the sides opposite them will be equal*; prop. 4 and 8 13 and 14 18 and 19 . . . 24 and 25, are respectively the converse of each other, and prop. 29 is the converse of prop. 27 and 28. See Ludlam's *Introduction to Euclid*.

In a proposition, the conditions laid down are called the *premises*, or more frequently the *hypothesis*. The word *hypothesis* in its general acceptation, means a *supposition on which reasoning is founded*, if an hypothesis be true all deductions from it will be true, but if the hypothesis be false the consequences drawn from it will be false likewise.

A direct

A direct or positive demonstration is that which gives a direct and certain proof of the proposition; it begins with the premises laid down in the proposition, and from these and self evident principles it deduces consequences, and from these and the principles, other conclusions, and so on, thus it proceeds deducing one truth from another in succession, till that which was at first to be proved evidently follows.

An indirect, negative or apagogical demonstration, called also a *reductio ad absurdum*, is that which proves a proposition, by shewing that the contrary supposition is absurd and impossible; accordingly it begins by assuming a proposition which directly contradicts what we mean to prove, and concludes by shewing the absurdity and falshood of the assumed proposition.

A proof by induction is such a one as arises from observing that a proposition is true in every particular case to which it has been applied, and from thence inferring that it holds true in all cases; such a proof, although it does

not amount to a general demonstration, is sometimes necessary when a general proof cannot be had.

* The solution or resolution of a problem is the process or system of reasoning, employed in finding the answer required; when this is performed by numbers, it is called a numerical solution; when by geometrical principles, a geometrical solution; and a mechanical solution when it is obtained by trials.

A lemma is a proposition previously laid down, as necessary to shorten the proof of some proposition which follows.

A corollary or consequence, is a truth obtained over and above what the proposition requires, and whose demonstration is for the most part included in that of the proposition.

* In a problem, the conditions laid down, including all the quantities given, are called the *data*, and the unknown quantity or answer the *quæsitæ*.

A scholium

A scholium is a remark or observation; made on something preceding.

Analysis is a method of investigating or discovering truth, by resolving a proposition into its original principles or component parts, so as to determine precisely what those parts or principles are; this is called the method of invention or resolution, or the method of reasoning *à posteriori*, employed in tracing and finding out causes from their effects, and is that made use of in Algebra.

Synthesis, or the synthetic method, is that by which truth is obtained, by laying down known principles, and pursuing the consequences flowing from them, till we arrive at a conclusion before unknown; this is called the method of composition, or of reasoning *à priori*, it proceeds from causes to find out and determine their effects, and is therefore the reverse of analysis; hence whatever has been found by analysis, may be demonstrated in a contrary order by synthesis; by this method the propositions in Euclid are demonstrated.

We now proceed to offer some advice to the unassisted learner respecting the order in which the Mathematical Sciences are to be acquired, with the least difficulty; and in this I would not be understood as dictating to those who have the advantage of tuition; the best method such can observe, if they desire to improve, is to follow strictly the advice of their tutor; nor would I insinuate that the following is the only successful plan, there may be others equally good; but experience has taught me that this is very well adapted to the generality of learners.

First, I would recommend an attention to the fundamental rules of * Arithmetic, viz. Nume-

* Hutton's Practical Arithmetic, and Walkingame's Tutors' Assistant, are proper books for this purpose.

In procuring elementary books, those are to be preferred which contain the whole subject from first principles, or which refer to Euclid's Elements, or some well-known book; by not observing this rule, it will oftentimes be necessary to procure several books before one subject can be understood—see instances of this in Emerson's Works, particularly in his Arithmetic of Infinites and Conic Sections.

ration,

ration, with addition, subtraction, multiplication, and division, both simple and compound; reduction, vulgar and decimal fractions, extraction of the square root, and a little of the rule of three; these will be sufficient, if understood well enough to perform the operations with tolerable ease; but it will not be necessary at this time for the learner to trouble himself about the reasons on which these operations are founded, they will be better understood hereafter, for it seldom happens that young persons can understand the theory till they are pretty expert in the practice.

The learner may now begin Algebra, taking care to acquaint himself well with notation and the use of the signs, and to understand practically the rules up to quadratic equations, omitting for the present their demonstrations, with whatever else appears difficult; it will now be time to undertake the * Elements of Euclid, learning the
defini-

* I make use of Simson's Euclid, thinking it the easiest for beginners. Mr. Playfair's Euclid is a very good one, and contains

definitions by heart, and if they are not all thoroughly understood he need not be uneasy, for the use and application which perpetually occur in the propositions, will perfect him in them.

Leaving the definitions, the learner will see, and readily acknowledge the possibility of what the postulates require, and admit the self-evident truths affirmed in the axioms; these, with the definitions and postulates, he must endeavour to remember, as being the principles from which every deduction is made. The 12th axiom may however be omitted for the present, as not being likely to be understood; but it must be referred to immediately after the 28th proposition in the first book, when its truth and evidence will appear.

tains some useful things which the former does not, as the rectification and quadrature of the circle, &c. but the 5th book, which is demonstrated in the concise language of Algebra, will not easily be understood without assistance. Watts's or Duncan's Logic should now be read, as affording much useful information on reasoning; and after these Watts's Improvement of the Mind, of which Dr. Johnson observes, that "Whoever has the care of instructing others, may be charged with deficiency in his duty, if this book is not recommended."

Johnson's Life of Dr. Watts.

In

In the propositions the general enunciation comes first, in this the premises are laid down, and the consequence intended to be proved is affirmed in general; the enunciation is next applied to the annexed figure; then follows the construction in which all lines, &c. necessary to the intended proof, are directed to be drawn; next the truth affirmed in the enunciation is demonstrated; and lastly, the enunciation is repeated with an affirmation that what was required is done or proved*.

The demonstration of a proposition is not to be learned by heart only, but the connection and agreement of every step with the preceding, or with some self-evident truth previously laid down, must be ascertained as we go on, so that every conclusion may be traced back to first principles, and its necessary dependence on them clearly seen; this will not merely procure the assent, but pro-

* Q E D, at the end of a demonstration, denotes *Quod erat demonstrandum*, which was to be demonstrated; Q E F *quod erat faciendum*, which was to be done; and Q E I *quod erat investigandum*, which was to be investigated.

duce the fullest conviction of its truth and certainty*.

In demonstrating the propositions in Euclid, if the principle or proposition on which any conclusion depends, be not perfectly remembered, reference must be made to it—for this purpose

* A learned French nobleman residing in this country, in a letter to one of the author's pupils, gives him the following excellent advice on the subject.

“ M. votre Pere m'a marqué, mon bon ami, que vous apprenez les Mathematiques j'en suis bien aise, parceque, de toutes les Sciences, c'est sans contredit, la plus nécessaire, dans quelque état qu'on se trouve.

“ Si vous voulez m'en croire, vous ne vous bornerez pas simplement à concevoir; mais vous vous habituerez à vous en démontrer les vérités, aussi parfaitement, que si vous vouliez les démontrer aux autres. C'est comme cela que cette science vous deviendra surtout utile.

“ Me pardonnez vous, mon Bon ami, de vous dire que je redoute pour votre curiosité? redoutez la également; et ne vous permettez, surtout en commençant, de marcher de vérités en vérités, qu'autant que vous les aurez bien approfondies. Les premiers principes vous paroîtront aisés; mais c'est surtout dans leur démonstration que git l'essentiel.”

notes are given in the margin, and by turning to the proposition there referred to, it must be examined and compared with the matter in hand, until their connection is clearly ascertained,

Having finished the sixth book, a review of what has been done may next take place; all the rules in * Arithmetic and Algebra which have been passed through, must be demonstrated; and whatever has been omitted must be carefully re-examined and understood. † Plane trigonometry comes next in order, and having learned the definitions, and demonstrated the propositions, it will be necessary, previous to entering on the practical part, to learn the use of a case of mathematical instruments, a little practical geometry, the nature and use of logarithms, with the use of

* Mr. Bonnycastle's Arithmetic contains the demonstrations of the rules, and is the best book for this purpose extant.

† I know of no single book which contains all that is necessary to the Theory and Practice of Plane Trigonometry, adapted to the capacity of beginners; however, it is my intention to publish a work on the subject which will contain Plane and Spherical Trigonometry, Logarithms, Tables, Practical Geometry, the Use of a Case of Instruments, &c. with whatever else is requisite to this most useful branch.

the

the tables of logarithms, sines and tangents; there are many pleasant and useful questions in plane trigonometry which may be solved by means of these, and the operations may be proved by the instruments.

If the learner intends to make navigation, fortification, surveying, or any other branch which depends on plane geometry, his particular study, he may stop here, or rather he may apply himself to that particular branch which will require little more than an application of the principles he has learned*.

But if solid Geometry be necessary, the eleventh and twelfth books of Euclid must be learned, and at intermediate times† the application of Algebra

* Robertson's Navigation, Muller's Fortification, and Bonnycastle's Mensuration, are the best books on these subjects in the English language.

A good little treatise on surveying will be found in Hutton's Mathematical Dictionary, vol. 2. p. 542.

† Simpson's Algebra will be found a very useful book for this purpose.

to plane Geometry, with the higher parts of Algebra, as cubic and biquadratic equations, infinite series, &c. It will likewise be necessary, now or even earlier, to enter on some elementary treatise of † Natural Philosophy and Astronomy; this may be considered as a relaxation, and will afford both pleasure and profit. Spherical Trigonometry with the projection of Spherical Triangles, on a plane, will be requisite for those who would enter deeply into Astronomy, as will likewise the † Conic Sections; § the application of Algebra to the Geometry of Curves, and lastly the doctrine of Fluxions†† and

† Nicholson's and Adams's Philosophy, are excellent introductions, but they are expensive. I have commonly used Hamilton's four Introductory Lectures to begin with, and afterwards Helsham's Philosophy. Bonnycastle's Astronomy is perhaps the best introduction in print.

† Vince's Conic Sections, or those in Ward's Young Mathematicians Guide, or in Hutton's Mathematics, may be used as introductions, but they do not enter far into the subject. We make use of Dr. Hamilton's.

§ This subject is treated of, at the end of Maclaurin's Algebra.

†† Simpson's Fluxions, and Emerson's Increments are proper books for this purpose.

and Increments; these have their application in almost every branch of science, and will be sufficient to exercise the utmost abilities of the learner, and to give full scope to his genius.

It has been often remarked, that in company, a meer scholar is a very insipid character; supposing this to be the case, what must a meer Mathematician be? It behoves the learner then to guard against those singularities, which too often characterize the studious, and to fill up his intervals of leisure in improving conversation with a chearful and judicious friend, and with such other studies, as will best improve his mind, and fit him for social intercourse. History is peculiarly adapted to this purpose, and that of our own country, should by no means be omitted. Ingenious and moral essays, such as those in the Spectator, Rambler, Adventurer, Guardian, &c. may be read with profit. And here may I presume to recommend a study, which for its superior importance demands our first regard, I mean *the study of the sacred scriptures, and a strict attention to the duties they inculcate.* A spirit of piety is so far from being inconsistent with true learning, (as some infidels would have us think) that it adds a lustre and dignity to all our performances. This is exemplified in the accounts we have of Newton, Locke, Hales, Marlborough, and recently in the conduct of the gallant Nelson. Surely from this consideration only, the advice will not be rejected, and there are others far more weighty. *The fear of the Lord is the beginning of wisdom.* Offer up therefore (says the learned and pious Dr. Watts) your daily requests to God, the father of lights, that he would bless all your attempts and labours in reading, study, and conversation. If the heathens in their works of learning thought it necessary to begin with God, is it not much more the indispensable duty of Christians to do so? Bishop Saunderson says, that study without prayer is Atheism.

CONTENTS.

	PAGE
D EFINITIONS AND NOTATION	1
Examples in Notation	8
A DDITION	11
S UBTRACTION	22
M ULTIPLICATION	26
D IVISION	35
F RACTIONS	53
Addition of Fractions	76
Subtraction of Fractions	79
Multiplication of Fractions	81
Division of Fractions	88
I NVOLUTION	97
Sir Isaac Newton's Rule	101
E VOOLUTION	105
To Extract the Square Root of Compound	
Quantities	109
To Extract the Roots of Powers in general	112
R EDUCTION OF SIMPLE EQUA-	
TIONS	117
Easy Problems containing one unknown	
Quantity	131
Reduction of Equations containing two un-	
known Quantities	142
e	Problems

	PAGE
Problems involving two unknown Quantities	150
Reduction of Equations, containing three unknown Quantities	158
REDUCTION OF AFFECTED QUA- DRATIC EQUATIONS	160
Problems producing Quadratic Equations	166
Surds	173
Addition of Surds	181
Subtraction of Surds	183
Multiplication of Surds	185
Division of Surds	187
Involution of Surds	189
Evolution of Surds	191
A Rule for finding Surd Multipliers	193
The method of removing a Quantity, from one term of a Fraction to the other, explained	196
INFINITE SERIES	197
The Binomial Theorem	198
A Rule for finding any Term in an infinite Series	205
A Rule to find the Sum of a Series	206
ARITHMETICAL PROPORTION	209
A Rule for finding any Number of Arithme- tical mean Proportionals between two given Quantities	212
	OF

CONTENTS.

xxix

PAGE

OF RATIO AND PROPORTION		213
The Composition of Ratios		216
The Resolution of Ratios		217
Theorems in Geometrical Proportion and Progression		218
A Rule for finding any Number of Geome- trical mean Proportionals		222
HARMONICAL PROPORTION		224
A GENERAL VIEW OF THE NATURE, FORMATION, AND ROOTS OF EQUATIONS		226
A Rule for depressing an Equation to lower Dimensions		230
THE RESOLUTION OF EQUATIONS OF SEVERAL DIMENSIONS BY APPROXIMATION		231
Extraction of the Roots of Pure Powers by Approximation		237
PROBLEMS WITH THE METHOD OF RESOLVING THEM POINTED OUT		240
GENERAL PROBLEMS		249
The method of Registering explained		253
Theorems in Simple Interest		257
Theorems in Discount		258
Theorems in Compound Interest		259

ERRATA.

To be corrected with the pen previous to entering on the subject.

Note. ex. signifies example, l. line, and b. from the bottom of the page.

- Page 6. l. 16. for $a+b-c$ read $a+b-c$
8. l. 7. - $x \times y$ - - - $x-y$
1. 13. - $a+x$ - - - $a+b$
10. ex. 4. - $24 \times 32 = 768$ - - - $10 \times 12 = 120$
16. ex. 7. in the answer dele $\circ$
19. ex. 8. for $2x+5y+x^2$ read $2x-5y+3x^2$
25. l. 1. b. insert $-a$
35. ex. 6. for $-x$ - read $-x$
39. l. 8. b. - greater - - - less
41. ex. 6. - $-9axy^3$ - - - $-9xy^3$
47. ex. 4. - $-b^2ad$ - - - $-b^2-2ad$
59. ex. 7. - $\frac{1-x}{4}$ - - - $\frac{1+x}{4}$
70. ex. 4. - ca - - - ca^2
71. ex. 9. - x^3-y^3 - - - x^2-y^2
79. ex. 13. anf. x^2+1 - - - x^2-1
96. ex. 9. ans. $8y^2$ - - - $4y^2$
97. ex. 10. anf. $27a+9a^2+16$ - - - $33a+16$
116. ex. 8. - $16x$ - - - $16x^4$, and for 4 read y^4
143. l. 2. b. - it equals - - - its equal
173. l. 1. - A's payment - - square of A's payment
177. ex. 1. anf. $12^{\frac{1}{2}} \sqrt[3]{\frac{1}{3}}$ - $12^{\frac{1}{2}} \sqrt[3]{\frac{1}{3}}$, and for $1728^{\frac{1}{2}}$ read $1728^{\frac{1}{3}}$
195. ex. 2. anf. $\frac{7}{5}$ - - - 7
216. l. 3. - 3, 4, 4, 8 - - - 3: 6, 4: 8
237. ex. 11. - x^3 - - - x^2
240. ex. 3. - nx^4 - - - x^4
261. l. 15. - 18 shillings - - 12 shillings
264. l. 8. - - dele from

ALGEBRA.

DEFINITIONS AND NOTATION.

ALGEBRA is a method of reasoning, applied in performing the calculations of quantities, by means of general characters.

The characters made use of, are the letters of the alphabet, $a b c d$, &c. the figures of common arithmetic 1 2 3 4, &c. and certain signs as $+ - \times \div$, &c.

The letters (to which the figures are sometimes joined) are used to denote quantities of all kinds; and in the resolution of problems the first letters $a b c d$, &c. are usually put for given quantities, whose values are known, and the last letters $x y z$ for unknown quantities, whose values are sought. The signs denote the operations to be performed.

1. The sign $+$ is the mark for addition; it is named *plus* (more), and denotes that the quantity to which it is prefixed must be added.

Thus, $a + b$, is read *a plus b*, and signifies that the quantity represented by b , is to be added to the quantity represented by a . Also, $2 + 3$ (*2 plus 3*) denotes the addition of the number 3 to the number 2.

2. The sign $-$ is the mark for subtraction; it is named *minus* (less), and signifies that the quantity to which it is prefixed must be subtracted.

Thus, $a - b$ is read *a minus b*, and denotes that the quantity represented by b , is to be subtracted from the

2 DEFINITIONS AND NOTATION.

quantity represented by a , also $5-2$ (5 minus 2) shews that 2 is to be subtracted from 5 .

3. Every algebraic quantity must have one of these signs either $+$ or $-$ belonging to it. Those quantities to which the sign $+$ is prefixed are called *positive* or *affirmative* quantities, and those with the sign $-$ are called *negative*, and note, that the sign always belongs to the quantity which immediately follows it, and to none else.

When an affirmative quantity stands by itself, or is the leading one, when connected with others, its sign $+$ is usually omitted; but the negative sign $-$ is never omitted in any case.

4. Quantities are said to have like signs, when the signs are all $+$ or all $-$, and to have unlike signs when some are $+$ and others $-$.

Thus $+2ax$, and $+3bc$ have like signs, so have $-4y$ and $-7ax$; but $3ay$ and $-4ay$ have unlike signs, as have $-3y$ and $4y$.

5. The sign $\times$ is the mark for multiplication; it is named *into*, and signifies that the quantities between which it is placed are to be multiplied together.

Thus $a \times b$ is read a into b , and denotes that the quantity represented by a must be multiplied into, or (as it is often improperly expressed) by the quantity b ; also, 3×4 (3 into 4) shews, that 3 is to be multiplied into 4 .

6. The sign of multiplication is not always expressed, for when a number and a letter, or several letters stand together like the letters of a word, then the product of these quantities is understood, though the sign $\times$ is not interposed.

Thus $3a$ denotes $3 \times a$, and ab denotes $a \times b$, and $4xyz$ is the same as $4 \times x \times y \times z$.

7. The sign $\div$ is the mark for division; it is named *by*, and denotes that the quantity which stands before it, is to be divided by that which follows.

DEFINITIONS AND NOTATION. 3

Thus $a \div b$ is read *a by b*, and shews, that the quantity *a* is to be divided by the quantity *b*; also, $12 \div 3$ (*12 by 3*) shews that 12 must be divided by 3.

The sign of division is frequently omitted, and instead of it, the dividend is placed above the divisor with a line between like a fraction.

Thus $a \div b$ is commonly written $\frac{a}{b}$, and $12 \div 3$ is the same with $\frac{12}{3}$ and $ab \div c$ is likewise written $\frac{ab}{c}$.

8. The sign $=$ is the mark of equality; it is expressed by the word *equals*, and denotes that the quantities between which it is placed are equal to each other.

Thus, $3 + 4 = 7$ is read 3 plus 4 equals 7, and shews that if 3 and 4 be added together, the sum is 7; also, $12 - 8 = 4$ (*12 minus 8 equals 4*) shews that 12 with 8 subtracted from it leaves a remainder of 4, in like manner $a + b = c - d$ (*a plus b equals c minus d*).

9. Two dots placed between quantities denote the ratio of those quantities to each other; thus $a : b$ expresses the ratio of *a* to *b*, and $5 : 4$ expresses that of 5 to 4.

10. Four dots thus $::$ denote proportion. The expression $a : b :: c : d$ denotes that *a* has the same ratio to *b* that *c* has to *d*, and is read thus, *a is to b as c to d*.

11. The coefficient of a quantity is the number prefixed to it.

Thus, in the quantities $3a$, $4ax$, $7xyz$, 3, 4, and 7, are the respective coefficients.

When a quantity consists of two or more letters, those letters may be considered as coefficients of each other.

Thus, in ab *a* is the coefficient of *b* and *b* of *a*, and in the quantity axy *a* may be considered as the coefficient of *xy*.

4 DEFINITIONS AND NOTATION.

cient of xy , x the coefficient of ay , and y of ax . These coefficients are sometimes called fellow factors; it must likewise be observed, that the numbers are called numeral coefficients, and the letters literal ones.

12. A simple quantity is that which consists of one letter, or of several joined together like the letters of a word, or of a number and letters so joined.

Thus, x , ay , $3ab$, $5axyz$, are simple quantities.

13. A compound quantity consists of several simple ones, called its terms, connected by the sign $+$ or $-$.

14. A binomial is a compound quantity consisting of two terms, a trinomial of three, a quadrinomial of four, and a multinomial of several terms.

Thus, $a+x$ is a binomial, $a+y+z$ a trinomial, $a-b+c-d$ a quadrinomial, and $a+b-c+d+x-3y$, &c. is a multinomial.

15. A residual quantity is a binomial, having one of its terms negative.

Thus, $a-b$, $x-y$, and $a-z$, are residuals.

16. The power of a quantity is either the quantity itself, called its first power, or it is the product which arises by multiplying the quantity into itself, which product is called its square or second power, or it is the product of this last into the given quantity, which product is called its cube, &c.

Thus, a is the 1st power of a .

$a \times a$ or aa is the 2d power or square of a .

$aa \times a$ or aaa is the 3d power or cube of a .

$aaa \times a$ or $aaaa$ is the 4th power or biquadrate of a , &c.

17. But these powers are commonly and more conveniently represented by figures placed over the letters and a little to the right, as a^2 , x^3 , y^4 , these figures are called indices or exponents.

Thus, x^2 is the same with xx , and represents the square of x .

x^3 is the same as xxx , and denotes the cube of x .

DEFINITIONS AND NOTATION.

5

x^4 is the same as $xxxx$, and denotes the 4th power of x ;

In like manner x^5 , x^6 , x^7 , &c. represent the 5th, 6th, 7th, &c. powers of x , where 2, 3, 4, 5, 6, and 7, are the indices respectively.

18. The root of a quantity is that which being multiplied continually into itself at length produces the given quantity.

Thus a is the 2d root or square root of a^2 , because $a \times a = a^2$; in like manner a is the cube root of a^3 , the 4th root of a^4 , &c.

19. The roots of quantities are denoted by the character $\sqrt{}$, called a radical sign, placed before the quantities, or more conveniently by fractional indices placed over them.

Thus, $\sqrt{a}$ or $a^{\frac{1}{2}}$ denotes the square root of a ;

$\sqrt[3]{a}$ or $a^{\frac{1}{3}}$ — its cube root;

$\sqrt[4]{a}$ or $a^{\frac{1}{4}}$ — its 4th root, &c.

20. Quantities which have no radical sign or fractional index, are called rational quantities.

21. Quantities which have a radical sign or fractional index, are called irrational quantities or surds.

22. Like quantities are such as consist of the same letters with the same indices.

Thus, a^2b^3 , and $4a^2b^3$, and $12a^2b^3$, are like quantities, so are $3xy^2 - 4xy^2$ and $-7xy^2$.

23. Unlike quantities are such as consist of different letters, or of the same letters with different indices.

Thus ax , bx , cx^2 , and b^2x , are unlike.

24. A vinculum is a straight line drawn over the top of a compound quantity, thus, $a+x-z$; it denotes that all the quantities under it are to be considered as one, with respect to any sign which connects them to other quantities.

6 DEFINITIONS AND NOTATION.

Thus, $\overline{a+b} \times c$ denotes that the sum of a and b is to be multiplied into c ; also, $\overline{3+4+5} \times 6$ shews that the sum of 3, 4, and 5, must be multiplied into 6. The following example is given both with and without the vinculum, to shew the error which would arise by omitting it when it should be used, or employing it when it should be omitted.

The product $\overline{9+2} \times 8 = 88$ where the vinculum is used.
 $9+2 \times 8 = 25$ where the vinculum is omitted.

So that the error which would arise by improperly omitting or using the vinculum in this example is 63.

In reading quantities under the vinculum the word both or all is used.

Thus, $\overline{x+y} \times z$ is read x plus y both into z , and $\overline{a-b} \div c$ is read a minus b both by c ; so also $\overline{a+b-c} \times d$ is read a plus b , minus c , all into d ; in like manner $\overline{x+y} \times \overline{a-b+c}$ is read x plus y both into a , minus b , plus c all; and $\overline{x+y-z} \div \overline{x-y}$ is read x plus y , minus z , all by x , minus y both.

25. The parenthesis is frequently used instead of a vinculum, thus, $\overline{a+b} \times c$ is written $(a+b) \times c$; and $\overline{x+y-a+b-c}$ is written $(x+y) \div (a+b-c)$.

When compound quantities under a vinculum are to be multiplied into others, the sign $\times$ of multiplication is sometimes omitted, and a dot used instead, thus, $\overline{a+b} \times \overline{c-d}$ is written $\overline{a+b.c-d}$, or $(a+b)(c-d)$; also, $\overline{3+x} \times \overline{4-y} \times \overline{5+z}$ is written $\overline{3+x.4-y.5+z}$; or thus, $(3+x).(4-y)(5+z)$.

Both the vinculum and parenthesis are often omitted in division: thus, $\overline{a+c} \div x$ may be written $\frac{a+c}{x}$; and

DEFINITIONS AND NOTATION. 7

$(a-x+z) \div (x+y)$ is written $\frac{a-x+z}{x+y}$; likewise, $(x+y+z+1) \div (x-y-z)$ is properly denoted by $\frac{x+y+z+1}{x-y-z}$.

The place of the vinculum is even sometimes supplied by two dots placed at each end of a compound quantity, and frequently at one end only.

Thus, $c \times \frac{x+y}{z} + 4$ may be written $c \times : \frac{x+y}{z} + 4$; or $c \times : \frac{x+y}{z} + 4$; also, $\sqrt{x+y}$ is written $\sqrt{ : x+y } ;$ or $\sqrt{ : x+y } ;$ and $\overline{x+y} \times \overline{x+y+z}$ may be written $: x+y : \times : x+y+z :$ or $x+y : \times : x+y+z :$.

26. The sign $>$ is called *greater than*, and the sign $<$ *less than*.

Thus, $a > b$ denotes that a is *greater than* b ; also, $b < a$ denotes that b is *less than* a .

27. The character ω is used to denote the general difference of two quantities when it is not known which is the greater.

Thus, $a \omega b$ expresses the difference of a and b which ever is the greater. The same thing is sometimes denoted thus, $\pm a \mp b$.

28. The reciprocal of a quantity is that quantity inverted, or unity (viz. 1) divided by it.

Thus, the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$; the reciprocal of $\frac{3}{4}$ is $\frac{4}{3}$; the reciprocal of a or its equal $\frac{a}{1}$ is $\frac{1}{a}$, and the reciprocal of $\frac{1}{3}$ is $\frac{3}{1}$ or 3.

The following brief abstract exhibits the substance of what has been said in one view.

8 DEFINITIONS AND NOTATION.

$+$ denotes addition. $-$ subtraction. $\times$ or $.$ multiplication. $\div$ division. $=$ is the sign of equality. $>$ is the sign of majority or greater than; and $<$ of minority or less than. $:::$ denote proportion. $\sqrt{}$ is a radical sign.

$4a^3$ is a simple quantity. 4 is its coefficient. 3 its index or exponent. $x+y$ is a binomial, $x \times y$ a residual.

$a+b$ represents the sum

$a-b$ — the difference

ab or $a \times b$ the product

$\frac{a}{b}$ or $a \div b$ the quotient

} of a and b .

$\overline{a+b}$ are quantities under a vinculum, and are the same as $(a+b)$.

$\overline{a-x} \times \overline{b+c}$ is the same as $(a-x) \cdot (b+c)$.

a^2 is the square of a , and $\sqrt{a}$ or $a^{\frac{1}{2}}$ is the square root of a .

a^3 is the cube of a , and $\sqrt[3]{a}$ or $a^{\frac{1}{3}}$ its cube root.

a^4 is the 4th power of a , and $\sqrt[4]{a}$ or $a^{\frac{1}{4}}$ its 4th root.

$\frac{x+y}{a}$ and $\overline{x+y} \div a$, and $(x+y) \div a$, are the same, as

are $3\sqrt{x-y}$, $3 \cdot \overline{x-y}^{\frac{1}{2}}$, and $3(x-y)^{\frac{1}{2}}$.

EXAMPLES IN NOTATION*.

The learner is required to express algebraically according to the foregoing notation, the following quantities.

* Although Notation is a necessary first step, or introduction to what follows, yet a perfect knowledge of the whole is by no means requisite for a meer beginner; whatever therefore appears difficult may be omitted for the present, and made the subject of future inquiry.

EXAMPLES IN NOTATION.

9

1. a plus b , minus d , plus x , minus y , equals c into z .
Ans. $a+b-d+x-y=cx$.

2. a into b , minus c into d , plus x into y , equals a plus b , both into c .
Ans. $ab-cd+xy=a+b \times c$.

3. a by b , minus c by d , plus x by y , minus z into a , equals 1000
Ans. $\frac{a}{b}-\frac{c}{d}+\frac{x}{y}-az=1000$.

4. a plus x both, into a minus x both.
Ans. $a+x \times a-x$, or $(a+x) \times (a-x)$, or $a+x.a-x$.

5. a plus b both by c , into a minus z both, by 4, equals a into z .
Ans. $\frac{a+b}{c} \times \frac{a-z}{4}$; or thus, $\frac{a+b}{c} \frac{a-z}{4}=az$.

6. x plus y , minus z all, into x plus y both, by x , minus y both, equals 48.
Ans. $x+y-z \times \frac{x+y}{x-y}$, or $(x+y-z) \cdot \frac{x+y}{x-y}=48$.

7. a , minus b into c , plus d both, by a .

$$\text{Ans. } a-b \times \frac{c+d}{a} \text{ or } a-b \cdot \frac{c+d}{a}$$

8. a plus x , minus y , plus a into x , equals 100.

Ans.

9. a minus z , plus the square of x , equals y , plus 40.

Ans.

10. a into b , plus c by d , minus x into y , plus z , equals 12, plus x .
Ans.

11. z plus y both, into z minus 4 both, plus x by y , equals a by b .
Ans.

12. a into z both by 5, into x plus y both by z , equals a into z .
Ans.

13. a minus y into x , plus a , minus 5, equals $3x$ by y minus 1 both.
Ans.

14. a into x , plus b into y , minus c into z , all by x , equals 41.
Ans.

15. The cube of a , plus the cube root of a , minus the 5th power of x , plus the 5th root of x , equals a plus x both, by y . *Ans.*

16. The 4th power of z plus the 5th root of y , minus the 6th power of a , plus the square root of b , equals a plus b both, into x minus y both. *Ans.*

17. The square root of x , by the square of y , plus the cube of a , by the cube root of b , equals 10 times x plus y both. *Ans.*

18. The square of x into x , minus y both, plus the square root of y , by a plus b both, equals 100. *Ans.*

19. Express the squares and square roots of the quantities a , b , and c . *Ans.*

20. Express the 2d, 3d, 4th, 5th, 6th, 7th, 8th, and 9th powers and roots of the quantities a and b . *Ans.*

In the following Examples the numeral value of each letter is given, from which the value of each example is to be computed.

Let $a=4$, $b=6$, $c=8$, $d=1$. Then will

1. $a+b-c+d=4+6-8+1=3$.

2. $a-b+3c-4d=4-6+24-4=18$.

3. $\frac{2a+3b+4c}{5a} = \frac{8+18+32}{20} = \frac{58}{20} = 2\frac{13}{10}$.

4. $a+b \times a+c=4+6 \times 4+8=24 \times 32=768$.

5. $\frac{a+b}{c} + \frac{a+b}{d} = \frac{4+6}{8} + \frac{4+6}{1} = \frac{10}{8} + 10 = 11\frac{1}{4}$.

6. Required the numeral value of $a+b-4c+12ad$. *Ans.* 26.

7. Required the value of $a+b-d+4$. *Ans.* 13.

8. Required the value of $a-b+d+4$. *Ans.* 3.

ADDITION.

71

9. Required the values of $ab+cd$ and $ab-cd$.
Ans. 32 and 16.
10. Required the value of $\frac{a-d}{c-b}$ into $\frac{c-b}{a-d}$. *Ans.* 6.
11. Required the value of $(a+b-c).(d+4)$.
Ans. 10.
12. What is the value of $b^2+2bc+c^2$. *Ans.* 196.
13. Find the value of $abc-abd+bcd$. *Ans.* 216.
14. Required the value of $\frac{2a+b}{c-d}$. *Ans.* 2.
15. What is the value of $ac+\frac{c}{a}-5a+10$?
Ans. 24.
16. What is the value of $a+b:xc-d$. *Ans.* 70.
17. Required the value of $2c+b.\frac{a+c}{b-4d}$. *Ans.* 132.
18. Required the value of $2.a+b+c.b-a^2.c+d^{\frac{1}{2}}$?
Ans. 432.
19. What is the value of $\frac{a+d^4}{ab-8^{\frac{1}{2}}}$ *Ans.* $312\frac{1}{2}$.

ADDITION

Teacheth to find or express the sums of algebraic quantities.

CASE I.

To add quantities which are like, and have like signs.

RULE.

Add all the coefficients together, and place the common sign before their sum, and the common letters after it.

ADDITION.

EXAMPLES.

1.	2.	3.
$+ 3a$	$- 2bx$	$2a+b$
$+ 5a$	$- 5bx$	$3a+b$
$+ 2a$	$- 7bx$	$4a+b$
$+ 10a$	$- 14bx$	$9a+3b$

* Example 1. The coefficients 3, 5, and 2 added together make 10, before which writing the common sign $+$, and after it the common letter a , the sum is $+10a$.

Example 2. The coefficients 2, 5, and 7 added together make 14, to which prefixing the common sign $-$, and adjoining the common letters bx , the sum is $-14bx$.

* It is very common for beginners, when working these examples, to object and say, I am told that the letters in algebra stand for quantities—pray what particular quantities do these represent? what am I to understand by them? The answer is, that though in the solution of a problem each letter is put for some particular quantity, yet here the case is different. The letters are of no value in themselves, nor are any values assigned them in the above examples; all that is here intended is to shew how to perform certain operations with certain characters, characters which are perfectly insignificant in themselves, but which may be made to represent quantities whenever it is necessary to do so.

The reasons on which the rule is founded are extremely obvious; for let a represent any thing whatever, then, as in example 1, $3a$, $5a$, and $2a$ added together will be $10a$, or 10 times the thing a represents; if a be a pound, $10a$ will be 10 pounds; if a expresses a shilling, $10a$ will express 10 shillings.

That the sum of any number of affirmative quantities is affirmative, and of any number of negative ones, negative, is evident, for since the sum is made up of the quantities added, it must be of the same nature with them; wherefore, if the quantities are $+$, the sum must be $+$; and if the quantities are $-$, the sum must be $-$ also.

ADDITION.

13

Example 3. Is done like the foregoing examples—the leading quantities, viz. those in the left-hand row, having no sign, + is understood; and the quantities in the second row, having no co-efficients, 1 is understood as the co-efficient of each.

$$\begin{array}{r}
 4. \\
 3ax - 2y \\
 4ax - 5y \\
 ax - 3y \\
 \hline
 8ax - 10y \\
 \hline
 \end{array}$$

$$\begin{array}{r}
 5. \\
 x + y - z \\
 2x + 3y - z \\
 4x + 2y - z \\
 \hline
 7x + 6y - 3z \\
 \hline
 \end{array}$$

$$\begin{array}{r}
 6. \\
 3a^2 - 2y^2 \\
 4a^2 - y^2 \\
 2a^2 - y^2 \\
 \hline
 9a^2 - 4y^2 \\
 \hline
 \end{array}$$

$$\begin{array}{r}
 7. \\
 3ax^{\frac{1}{2}} - abc - 4 \\
 2ax^{\frac{1}{2}} - 3abc - 8 \\
 4ax^{\frac{1}{2}} - 2abc - 7 \\
 \hline
 9ax^{\frac{1}{2}} - 6abc - 19 \\
 \hline
 \end{array}$$

$$\begin{array}{r}
 8. \\
 2a \\
 3a \\
 9a \\
 \hline
 \hline
 \hline
 \end{array}$$

$$\begin{array}{r}
 9. \\
 -4ax \\
 -3ax \\
 -11ax \\
 \hline
 \hline
 \hline
 \end{array}$$

$$\begin{array}{r}
 10. \\
 2\sqrt{ax} \\
 3\sqrt{ax} \\
 5\sqrt{ax} \\
 \hline
 10\sqrt{ax} \\
 \hline
 \end{array}$$

$$\begin{array}{r}
 11. \\
 bac - ax \\
 abc - ax \\
 2abc - ax \\
 3abc - 3ax \\
 \hline
 \hline
 \hline
 \end{array}$$

$$\begin{array}{r}
 12. \\
 4a - \sqrt{x+y} \\
 2a - 3\sqrt{x+y} \\
 3a - 4\sqrt{x+y} \\
 7a - 2\sqrt{x+y} \\
 \hline
 \hline
 \hline
 \end{array}$$

$$\begin{array}{r}
 13. \\
 a + b - c \\
 3a + 2b - 4c \\
 4a + 5b - 3c \\
 2a + b - c \\
 \hline
 \hline
 \hline
 \end{array}$$

ADDITION.

14.

$$\begin{array}{r}
 -a^3 + xyz + 4 \\
 -a^3 + xyz + 12 \\
 -3a^3 + xyz + 1 \\
 -6a^3 + 4xyz + 17 \\
 \hline
 \hline
 \hline
 \end{array}$$

15.

$$\begin{array}{r}
 az - z^4 \\
 2az - z^4 \\
 3az - z^4 \\
 9az - z^4 \\
 \hline
 \hline
 \hline
 \end{array}$$

16.

$$\begin{array}{r}
 2y - 3\sqrt{x} \\
 4y - 5\sqrt{x} \\
 6y - 2\sqrt{x} \\
 7y - 3\sqrt{x} \\
 \hline
 \hline
 \hline
 \end{array}$$

CASE II.

To add quantities which are like, but have unlike signs.

RULE*.

Add all the affirmative co-efficients into one sum, and the negative ones into another.

* The truth of this rule will sufficiently appear by considering the nature of the quantities which are to be added. The addition here consists in finding what will arise by taking or incorporating several quantities together; some of which are additious, and others subductions; as such quantities are, in their effects, contrary, and destroy each other in proportion, as the lesser quantity prevails, their sum will be truly estimated, by considering how much the greater exceeds the less; that is, by taking the less from the greater, and prefixing the sign of the greater to what remains.

Suppose, in example 1, a denotes a shilling, and the quantities, with the sign $+$, are sums which I possess; then will those, with the sign $-$, be sums which I owe, being in effect contrary to the former; and let the example above referred to be considered as an account which I have to sum up or settle: now it appears that I have $3a$ and $5a$, or $8a$, that is, 8 shillings in my possession, and that I owe $4a$ or 4 shillings—this debt will certainly reduce my stock instead of increasing it; therefore, instead of adding it to the 8 shillings, I must deduct, and the remainder, $4a$ or 4 shillings, will be what I am worth.

“It may seem a paradox,” says Mr. Ludlam, “that what is called addition in algebra, should sometimes mean addition, and sometimes subtraction: but the paradox wholly arises from the scantiness of the name given to the algebraic process; from
“employing

ADDITION.

75

Subtract the less of these sums from the greater—
to the remainder prefix the sign of the greater, and
subjoin the common letters as before.

EXAMPLES.

1.	2.	3.
$+ 3a$	$11ax$	$axy + 3$
$- 2a$	$- 12ax$	$- 2axy - 4$
$+ 5a$	$7ax$	$- 3axy + 8$
$- 2a$	$- 9ax$	$7axy - 1$
$+ 4a$	$- 3ax$	$3axy + 6$

Explanation.

Example 1. The affirmative coefficients $+3$ and $+5$ being added together the sum is $+8$; and the negative ones -2 and -2 , added in like manner, the sum is -4 ; and subtracting the less of these from the greater, viz. 4 from 8, the remainder is 4; to which prefixing the sign $+$ of the greater, and annexing the common letter a , the sum is $+4a$.

Example 2. Here the sum of the affirmative coefficients 11 and 7 is 18; and the sum of the negative ones -12 and -9 is -21 ; from which subtracting the former sum 18, the remainder is 3; to which annexing the sign of the greater $-$, and the common letters ax , the sum is $-3ax$.

“employing an old term in a new and more enlarged sense. Instead of addition, call it incorporation, union, striking a balance, or any name to which a more extensive idea may be annexed than that which is usually implied by the word addition; and the paradox vanishes.”

Rudiments of Mathematics. p. 29.

4.

$$\begin{array}{r}
 3x^2 + ay \\
 -2x^2 - ay \\
 \hline
 4x^2 + 3ay \\
 -2x^2 + 4ay \\
 \hline
 2x^2 + 7ay \\
 \hline
 \end{array}$$

5.

$$\begin{array}{r}
 3\sqrt{x-y} \\
 -2\sqrt{x+y} \\
 \hline
 11\sqrt{x-3y} \\
 -\sqrt{x-y} \\
 \hline
 10\sqrt{x-4y} \\
 \hline
 \end{array}$$

6.

$$\begin{array}{r}
 -3\sqrt{y-8} \\
 -4\sqrt{y+4} \\
 \hline
 3\sqrt{y-8} \\
 -2\sqrt{y-10} \\
 \hline
 -6\sqrt{y-15} \\
 \hline
 \end{array}$$

7.

$$\begin{array}{r}
 10ax + 5x^{\frac{1}{3}} - 10 \\
 3ax - 4x^{\frac{1}{3}} - 4 \\
 -14ax + 5x^{\frac{1}{3}} + 18 \\
 2ax - 7x^{\frac{1}{3}} - 12 \\
 -ax + 6x^{\frac{1}{3}} + 8 \\
 \hline
 0 \\
 \hline
 \end{array}$$

8.

$$\begin{array}{r}
 4a \\
 -2a \\
 -3a \\
 -5a \\
 8a \\
 \hline
 \hline
 \hline
 \hline
 \hline
 \hline
 \end{array}$$

9.

$$\begin{array}{r}
 3ax \\
 -2ax \\
 4ax \\
 -7ax \\
 5ax \\
 \hline
 \hline
 \hline
 \hline
 \hline
 \hline
 \end{array}$$

10.

$$\begin{array}{r}
 ab - bc \\
 -ab + bc \\
 3ab - 2bc \\
 5ab - 4bc \\
 -2ab + 7bc \\
 \hline
 \hline
 \hline
 \hline
 \hline
 \hline
 \end{array}$$

11.

$$\begin{array}{r}
 x^2 - \sqrt{x} \\
 2x^2 + 3\sqrt{x} \\
 -7x^2 - 2\sqrt{x} \\
 4x^2 + 4\sqrt{x} \\
 -3x^2 - 2\sqrt{x} \\
 \hline
 \hline
 \hline
 \hline
 \hline
 \hline
 \end{array}$$

12.

$$\begin{array}{r}
 -3ax - 14 \\
 5ax + 20 \\
 -2ax - 2 \\
 7ax + 7 \\
 -4ax - 4 \\
 \hline
 \hline
 \hline
 \hline
 \hline
 \hline
 \end{array}$$

13.

$$\begin{array}{r}
 a - 8 \\
 2a + 3b \\
 4a + 5b \\
 3a - 7b \\
 4a - 9b \\
 \hline
 \hline
 \hline
 \hline
 \hline
 \hline
 \end{array}$$

ADDITION.

17

14.	15.	16.
$-ab - b^2 + 3c$	$a\sqrt{x} - ay^3$	$7axy - ax^{\frac{1}{3}}$
$2ab - 2b^2 - 4c$	$-3a\sqrt{x} + 2ay^3$	$-4axy + 2ax^{\frac{1}{3}}$
$3ab + 4b^2 - 5c$	$4a\sqrt{x} - 3ay^3$	$5axy - 7ax^{\frac{1}{3}}$
$-7ab - 3b^2 + 2c$	$8a\sqrt{x} + 9ay^3$	$-2axy + 7ax^{\frac{1}{3}}$
$4ab + 3b^2 - 3c$	$-12a\sqrt{x} + 2ay^3$	$9axy - 9ax^{\frac{1}{3}}$

CASE III.

To add quantities which are unlike, and have unlike signs.

RULE.*

Collect all the like quantities of one sort into one sum, by the former rules; and the like quantities of another sort, into another sum; and the like quantities of a third sort into a third sum; and so on till all the quantities are collected.

Set down these several sums, with their proper signs prefixed, as in the last rule.

* To understand the reason of this rule, it must be remembered that though we can add things of the same kind together, yet we cannot do so with things which are not of the same kind: thus, 2 shillings can be added to 2 shillings, and the sum is 4 shillings—this is evident enough—but 2 shillings cannot be added to 2 yards, so as to make either 4 shillings or 4 yards—here the actual addition is manifestly impossible, and all that can be done is to exhibit them in one point of view—this is exactly the method observed in the above examples—all the quantities which are alike are collected, and the sums set down; and the others, which cannot be incorporated, are set down with their proper signs, this is all that can be done, when the quantities to be added are of different kinds.

ADDITION.

EXAMPLES.

$$\begin{array}{r}
 1. \\
 3a \\
 -4b \\
 -2a \\
 +6b \\
 \hline
 a+2b \\
 \hline
 \end{array}$$

$$\begin{array}{r}
 2. \\
 4x \\
 5y \\
 3x \\
 -2y \\
 \hline
 7x+3y \\
 \hline
 \end{array}$$

$$\begin{array}{r}
 3. \\
 2a^2-b \\
 -3a+c \\
 -2a-b \\
 3a+4c \\
 \hline
 2a^2-2a-2b+5c \\
 \hline
 \end{array}$$

Explanation.

Example 1. Here $3a$ and $-2a$ being like quantities, their sum, by case 2, is a . Also, $-4b$ and $+6b$, being added by the same rule, their sum is $+2b$, these connected, the sum is $a+2b$.

Example 2. Here $4x$ added to $3x$ give $7x$, by case 1; and $5y$ added to $-2y$ give $+3y$ by case 2; these connected give $7x+3y$ for the sum.

Example 3. In this example $2a^2$ is the only quantity of that sort, it must therefore be put down. Next, $-3a$, $-2a$, and $+3a$, collected by case 2, give $-2a$ for their sum. Also, $-b$ and $-b$ give $-2b$, and $+c$ added to $+4c$ give $+5c$; these sums connected by their proper signs will be $2a^2-2a-2b+5c$, as in the example.

$$\begin{array}{r}
 4. \\
 5y \\
 -4x \\
 3x \\
 -2y \\
 -2x \\
 \hline
 3y-4x+x \\
 \hline
 \end{array}$$

$$\begin{array}{r}
 5. \\
 4ax-y \\
 3ay-2 \\
 -2ax+ay \\
 -7y+8 \\
 2ay+y \\
 \hline
 2ax+6ay-7y+6 \\
 \hline
 \end{array}$$

$$\begin{array}{r}
 6. \\
 a\sqrt{x}-11 \\
 3y+18 \\
 -19-3y \\
 2a\sqrt{x}+12 \\
 b-x \\
 \hline
 3a\sqrt{x}+b-x \\
 \hline
 \end{array}$$

ADDITION.

19.

$$\begin{array}{r}
 7. \\
 ax - ay^2 \\
 - 3ay + 5ax \\
 - 2ay + 7ay \\
 - 4ax - 8ay^2 \\
 \hline
 2ax + 2ay - 9ay^2 \\
 \hline
 \end{array}$$

$$\begin{array}{r}
 8. \\
 x + x^2 \\
 2x - y \\
 3y - x \\
 2x^2 + 3y \\
 \hline
 2x + 5y + x^2 \\
 \hline
 \end{array}$$

$$\begin{array}{r}
 9. \\
 ay - x + 4 \\
 x + 3b - ay \\
 6 + x^2 - 4b \\
 b - 10 - x^2 \\
 \hline
 0 \\
 \hline
 \end{array}$$

$$\begin{array}{r}
 10. \\
 3a \\
 - 5b \\
 - 4a \\
 - 2b \\
 \hline
 \hline
 \end{array}$$

$$\begin{array}{r}
 11. \\
 2d \\
 3c \\
 4b \\
 5c \\
 \hline
 \hline
 \end{array}$$

$$\begin{array}{r}
 12. \\
 2ax \\
 - 3ay \\
 - ax \\
 - x \\
 \hline
 \hline
 \end{array}$$

$$\begin{array}{r}
 13. \\
 2b - c \\
 3d - b \\
 4c + 2d \\
 - b - 5d \\
 \hline
 \hline
 \end{array}$$

$$\begin{array}{r}
 14. \\
 ax - ay \\
 ax + ax \\
 ay - ax \\
 ay + 7ax \\
 \hline
 \hline
 \end{array}$$

$$\begin{array}{r}
 15. \\
 a - b \\
 c + d \\
 3a + b \\
 - 2c - 4a \\
 \hline
 \hline
 \end{array}$$

$$\begin{array}{r}
 16. \\
 2a - 10 \\
 3b + a \\
 4c - 3 \\
 11ax + b \\
 \hline
 \hline
 \end{array}$$

$$\begin{array}{r}
 17. \\
 a^2 + b \\
 x - 12 \\
 ax - x^2 \\
 4x^3 - \sqrt{x} \\
 \hline
 \hline
 \end{array}$$

$$\begin{array}{r}
 18. \\
 a^{\frac{1}{2}} - y \\
 \sqrt{x} + 3\sqrt{y} \\
 y - \sqrt{x} \\
 4\sqrt{y} - a^{\frac{1}{2}} \\
 \hline
 \hline
 \end{array}$$

Of Quantities under a Vinculum.

Quantities under a vinculum are added together like others, by their coefficients—all the quantities connected by the vinculum must be considered as one simple quantity, and the adjoining quantity as its coefficient; thus, $3.\overline{x+y}$ added to $5.\overline{x+y}$, the sum is $8.\overline{x+y}$, where 3 and 8 are the coefficients of $\overline{x+y}$.

EXAMPLES.

1.

$$\begin{array}{r}
 3\sqrt{a-y} \\
 4\sqrt{a-y} \\
 2\sqrt{a-y} \\
 7\sqrt{a-y} \\
 \hline
 16\sqrt{a-y}
 \end{array}$$

2.

$$\begin{array}{r}
 -4.\overline{a+b} \\
 3.\overline{a+b} \\
 -2.\overline{a+b} \\
 7.\overline{a+b} \\
 \hline
 4.\overline{a+b}
 \end{array}$$

3.

$$\begin{array}{r}
 3(x+y)^2 \\
 4(x-y)^{\frac{1}{2}} \\
 -3(a-b) \\
 4(x+y)^2 \\
 \hline
 7(x+y)^2 + 4(x-y)^{\frac{1}{2}} - 3(a-b)
 \end{array}$$

Example 1 is done by case 1. Example 2 by case 2; and example 3 by case 3.

ADDITION.

21

$$\begin{array}{r}
 4. \\
 \hline
 3. \overline{x-y} \\
 4. \overline{x+y} \\
 -3. \overline{x-y} \\
 -2. \overline{x+y} \\
 \hline
 2. \overline{x+y} \\
 \hline
 \end{array}$$

$$\begin{array}{r}
 5. \\
 \hline
 2\sqrt{ab+x} \\
 3\sqrt{ax+b} \\
 -4. \overline{x-y} \\
 -2\sqrt{ab+x} \\
 \hline
 3\sqrt{ax+b} - 4. \overline{x-y} \\
 \hline
 \end{array}$$

$$\begin{array}{r}
 6. \\
 \hline
 3. \overline{a+b} \\
 -4. \overline{x-y} \\
 2\sqrt{x+z} \\
 3. \overline{a+b} \\
 \hline
 \hline
 \hline
 \end{array}$$

$$\begin{array}{r}
 7. \\
 \hline
 2\sqrt{y-4} \\
 3\sqrt{y-4} \\
 12. \overline{x-y} \\
 -4. \overline{x-y} \\
 \hline
 \hline
 \hline
 \end{array}$$

PROMISCUOUS EXAMPLES FOR PRACTICE.

In the following examples the proper arrangement of the quantities is left to the learner; the best method is to place like quantities under each other when they occur.

1. Add $3a+4b$, and $2b-3c$, and $3c-b+a$ together.

These quantities, properly arranged, will stand thus—

$$\begin{array}{r}
 3a+4b \\
 2b-3c \\
 a-b+3c \\
 \hline
 4a+5b \\
 \hline
 \end{array}$$

and their sum

SUBTRACTION.

2. Add $a+b$, and $3a-c$, and $4a-3c+b$ together.
Sum $8a+2b-4c$.
3. Add $5x+5y$ and $3x-2y$ together. Sum $8x-3y$.
4. Add $b+a$ and $3a-5b$ together. Sum $4a-4b$.
5. To $4x-a$ add $3a-x$. Sum $3x+2a$.
6. To $4a-5b-10$ add $2y-x+4$.
Sum $4a-5b+2y-x-6$.
7. Add $x+z$ and $x-z$ together. Sum $2x$.
8. Add $a+b$, and $3b-x$, and $4x+3a-y$ together.
9. Add $x-4$, and $10-7x$, and $a-b-10$ together.
10. Add $3a-b+4c$, and $2x-c+4a$, and $5c+7b$ together.

SUBTRACTION.

Subtraction is the reverse of addition, and teacheth to find or express the difference of algebraic quantities.

RULE*.

Set those quantities from which the subtraction is to be made in one line, and the quantities to be sub-

* This rule being the reverse of addition, the method of operation must be so likewise; it depends on this principle, that to subtract an affirmative quantity is the same thing as to add a negative one; and to subtract a negative is the same as to add an affirmative; the truth of which will appear, by considering the nature of the quantities and of the operation: if we suppose affirmative quantities to represent property, then will negative quantities represent debts; now to take away (or subtract) any part of my property, is just the same as though you add a debt of equal amount; and to take away a debt from my account, is, in effect, the same as to increase my property.

That the difference of quantities should be sometimes found by addition, and sometimes by subtraction, is no paradox, but arises from the nature of those quantities; in finding the difference of latitude of two points on the globe, the same method is followed, we add when the points are on different sides of the equator, and subtract when they are both on the same side.

SUBTRACTION.

23

tracted in a line below them; observing to place like quantities under each other when they occur.

Change the signs of all the quantities in the lower line, or conceive them to be changed, and then add them (so changed) to the upper line, by the rules in addition, the result will be the difference required.

EXAMPLES.

1.	2.	3.
$3a-4b$	$3b-x+2z$	$x+y-4$
$2a-6b$	$-4b+2x-3z$	$x-y+5-z$
$a+2b$	$7b-3x+5z$	$2y-9+z$

Explanation.

Example 1. Here $2a-6b$ are the quantities to be subtracted, conceiving their signs changed, they become $-2a+6b$, this added to the upper line $3a-4b$, by case 2 in addition, the result is $a+2b$ as above.

Example 2. In like manner, if we suppose the signs in the lower line changed, it becomes $+4b-2x+3z$, this added to the upper line by case 2, gives $7b-3x+5z$.

Example 3. Here also, $x-y+5-z$, by changing signs, becomes $-x+y-5+z$, this added to the upper line $x+y-4$ gives $2y-9+z$, as in the example.

4.	5.	6.	7.
$3ay$	$4xz$	$-3a+2b$	$3y-1$
ay	$-2xz$	$4a+3b$	$-4z+x$
$2ay$	$6xz$	$-7a-b$	$3y-1+4z-x$

SUBTRACTION.

8.

$$\begin{array}{r}
 5x^2 - 6 - y \\
 6x^2 + 6 + y - \sqrt{x} \\
 \hline
 -x^2 - 12 - 2y + \sqrt{x} \\
 \hline
 \end{array}$$

9.

$$\begin{array}{r}
 a + b + c \\
 2a - b - c \\
 \hline
 \hline
 \end{array}$$

10.

$$\begin{array}{r}
 3ax - 4y \\
 2ax - z + 4 \\
 \hline
 \hline
 \end{array}$$

11.

$$\begin{array}{r}
 a + x - 1 \\
 3a - 2x + 4 \\
 \hline
 \hline
 \end{array}$$

12.

$$\begin{array}{r}
 2a\sqrt{x} - y \\
 -a\sqrt{x} + 40 \\
 \hline
 \hline
 \end{array}$$

13.

$$\begin{array}{r}
 x^3 + x^2 + x \\
 -x^3 + 3x^2 - 4x + 4 \\
 \hline
 \hline
 \end{array}$$

14.

$$\begin{array}{r}
 3a + 4 \\
 5a + 3 \\
 \hline
 \hline
 \end{array}$$

15.

$$\begin{array}{r}
 z + 12 - ax + y \\
 -z + 48 - ax + 3y \\
 \hline
 \hline
 \end{array}$$

16.

$$\begin{array}{r}
 -ax^2 - 8 + x^{\frac{3}{2}} \\
 -ax + 8 + 3 \\
 \hline
 \hline
 \end{array}$$

17.

$$\begin{array}{r}
 a + b - c + 4x \\
 a - b + 3c - z \\
 \hline
 \hline
 \end{array}$$

18.

$$\begin{array}{r}
 -4xy - a + ab - z^3 \\
 2xy - b + 14x + z^2 \\
 \hline
 \hline
 \end{array}$$

SUBTRACTION.

25

Quantities under a vinculum are subtracted by means of their coefficients, thus—

1. $\begin{array}{r} 3\sqrt{x+y} \\ -4\sqrt{x+y} \\ \hline 7\sqrt{x+y} \end{array}$	2. $\begin{array}{r} 4\sqrt{z-2}+3\sqrt{x-y} \\ 2\sqrt{z-2}-4\sqrt{x-y} \\ \hline 2\sqrt{z-2}+7\sqrt{x-y} \end{array}$
3. $\begin{array}{r} 2.(a+b)-x \\ 3.(a+b)-x^2 \\ \hline -(a+b)+x^2-x \end{array}$	4. $\begin{array}{r} 3\sqrt{a-x+y} \\ -4(a+b) \\ \hline \end{array}$

EXAMPLES FOR PRACTICE.

1. From $2x+3y$ take $3x-5y$. *Ans.* $-x+8y$.
2. From $x+z$ take $x-z$. *Ans.* $2z$.
3. From $x+3y$ take $3x-4y$. *Ans.* $-2x+7y$.
4. From $5a^2-2x$ take $2a^2-7x$. *Ans.* $3a^2+5x$.
5. From $4xy-5$ take $-xy+5$. *Ans.* $5xy-10$.
6. From $4\sqrt{a-4}+3x$ take $2\sqrt{a+1}-4x$. *Ans.* $2\sqrt{a-5}+7x$.
7. From $3a-4b$ take $2a-x+y-3z$. *Ans.* $a-4b+x-y+3z$.
8. From $a+b$ take $c-d+x-y$. *Ans.* $a+b-c+d-x+y$.
9. From $4\sqrt{a+b}$ take $\sqrt{a+b}-3\sqrt{x-y}$. *Ans.* $3\sqrt{a+b}+3\sqrt{x-y}$.
10. From $3ax-x^2+z^2$ take $a+z^2+\sqrt{a-1}$. *Ans.* $3ax-x^2-\sqrt{a-1}$.

11. From $a\sqrt{a-4}$ take $2a\sqrt{a-4}-108$.

Ans. $-a\sqrt{a-4}+108$.

12. From $a+b-c-d$ take $a+b-y-z$.

Ans. $-c-d+y+z$.

MULTIPLICATION.

Multiplication teaches to find the amount of any quantity, called the multiplicand, taken as many times as there are units in another quantity, called the multiplier.

The multiplicand and multiplier are called terms or factors, and the quantity which arises from the operation is called the factum or product.

CASE I.

When both factors are simple quantities.

RULE I.

Multiply the co-efficients of both factors together, and join all the letters in them to the product, the result will be the product required.

* When the signs of both factors are alike, the sign of the product must be +, but when the signs of the factors are unlike, that of the product must be —.

* That like signs in the factors give +, and unlike signs — in the product, will evidently appear from the following considerations.

1. When $+a$ is to be multiplied into $+b$, it is implied that a is to be taken as many times as there are units in b ; and since the sum of any number of affirmative terms is affirmative, it follows, that $+a$ taken b times is affirmative; that is, $+a \times +b$ gives $+ab$.

2. When quantities are multiplied together, it is indifferent in what order they are placed; for $a \times b$ is the same as $b \times a$; therefore,

MULTIPLICATION.

27

It is best, for the sake of method, first to determine the sign, then the coefficient, and lastly, the literal part of the product, and to place the letters alphabetically.

fore, when $-a$ is to be multiplied into $+b$, or $+b$ into $-a$: this is the same thing as taking $-a$ as many times as there are units in $+b$; and since the sum of any number of negative terms is negative, $-a$ times $+b$, or $+b$ times $-a$ will be negative; that is, $-a \times +b$ or $+a \times -b$ produces $-ab$.

3. When $-a$ is to be multiplied into $-b$, it is implied that $-a$ is to be (not added, but) subtracted as often as there are units in b , because the sign $-$ denotes subtraction; but subtracting a negative is the same as adding an affirmative, consequently, $-a$ subtracted b times is the same as $+a$ added b times, that is, $-a \times -b$ is the same with $+a \times +b$, which produces $+ab$, as has been shewn.

The substance of this note was taken from Mr. Ludlam's Rudiments, but the same thing may be demonstrated in another manner, which will, perhaps, be better understood by some.

If $a-a$, which is equal to 0 (nothing), be multiplied into any number as b , it is evident that the product will be 0. Let therefore $a-a=0$ be multiplied into b ; now the first term of the product will evidently be $+ab$ by case 1, therefore the second term must be $-ab$, otherwise the product will not be 0; consequently, $-a \times +b = -ab$, or $-$ into $+$, produces $-$.

Let now $a-a=0$ be multiplied into $-b$, the first term of the product being $-ab$, from what has been just now shewn; the second term must, of course, be $+ab$ to make the product $=0$; therefore, $-a \times -b = +ab$, or $-$ into $-$ produces $+$.

That these conclusions are true, may likewise be shewn by numbers, thus, let $7-3$ be multiplied into $6-4$, observing the rule given above for the signs, the work will stand thus—

$$\begin{array}{r} 7-3 \\ 6-4 \\ \hline 42-18 \\ -28+12 \\ \hline \end{array}$$

The product is $42-46+12=8$

Now, if instead of $7-3$ and $6-4$, their equals 4 and 2 be multiplied together, the product is likewise 8; the rule is therefore true in this instance, and will be found so in every case to which it can be applied.

D 2

MULTIPLICATION.

EXAMPLES.

1.	2.	3.	4.
$+4x$	$-5ax$	$+3y$	$-9ax$
$+3a$	$-4z$	$-4z$	$+4bc$
$+12ax$	$+20axz$	$-12yz$	$-36abcx$

Explanation.

Example 1. First, the signs of the factors being alike, (viz. both $+$) that of the product must be $+$; next the coefficients 3 and 4 multiplied together give 12, to which annexing the letters a and x , the product is $+12ax$.

Example 2. Here again the signs are alike (both $-$), therefore the sign of the product must be $+$, and the coefficients multiplied together give 20, and the letters are ax and z , these joined will make the product $+20axz$.

Example 3 and 4. In each of these examples the signs are unlike, therefore those of the products must be $-$; the coefficients and letters are found as in examples 1 and 2.

5.	6.	7.	8.
$ab\sqrt{x}$	xy	$-ay^2$	$-7x^3yz^2$
c	$-z$	$+x$	$-4ab$
$abc\sqrt{x}$	$-xyz$	$-4axy^2$	$+28abx^3yz^2$

9.	10.	11.	12.
$3ax$	$4ab$	$-5a^2$	$-13ab^{\frac{1}{2}}$
$4y$	$-3c$	$-7ax$	$2x^2$

MULTIPLICATION.

29

13.	14.	15.	16.
$-4axy$	$2xy^2$	$-abc$	$7axz$
$-cx$	$3ax$	xyz	$3bcd$

RULE 2.

When the same letter occurs in both factors, the multiplication is performed by adding the index of that letter in one factor, to the index of the same letter in the other factor, and placing the sum over that letter in the product.

If a letter have no index expressed, 1 is understood to be its index.

EXAMPLES.

1.	2.	3.	4.
a^2	$3a^3$	$-4x^2y^3z^4$	$3a^2bc$
a^3	$-a^4$	$-3x y z$	$4a^2bc$
a^5	$-3a^7$	$12x^3y^4z^5$	$12a^4b^2c^2$

Explanation.

Example 1. Here the indices 3 and 2 added together are 5, which placed as an index over the common letter a , the product is a^5 .

Example 2. The sign, and numeral part of the product are found as before directed, and the indices added together give 7, which placed over the common letter, the product is $-3a^7$.

MULTIPLICATION.

5. $3a^2y$ $4a^3x$ $12a^5yz$ 	6. $-4ax$ $3ay$ $-12a^2xy$ 	7. $-5ab^2$ $-3bc$ $15ab^3c$ 	8. axy^2 $-a^2by^3$ $-a^3bxy^5$
9. $4ax$ $5ayz$ 	10. $-4a^2x^2$ $-2a^2x^3$ 	11. $-axy^2$ $-a^2xy^3$ 	12. $5ax$ $-2a^3y$
13. $-7x^4$ $4a^2x$ 	14. $2a^2x^3y^4$ $3axyx$ 	15. $-9xy^2z^3$ $4x^3yz^4$ 	16. $-3a^2b^3c^4$ $-a^4b^3c^2$

CASE II.

When one of the factors is a compound quantity.

RULE.

Multiply every term of the compound quantity separately, into the simple one, as in case I, and place the products one after another with their proper signs.

MULTIPLICATION.

31

EXAMPLES.

$$\begin{array}{r} 1. \\ 2x-3y \\ 4a \\ \hline 8ax-12ay \end{array}$$

$$\begin{array}{r} 2. \\ 7a-3b \\ 2ac \\ \hline 14a^2c-6abc \end{array}$$

$$\begin{array}{r} 3. \\ a+5 \\ -b \\ \hline -ab-5b \end{array}$$

Explanation.

Example 1. Here $2x$ multiplied into $4a$ produces $8ax$, and $-3y$ into $4a$ produces $-12ay$, and these placed in order give $8ax-12ay$ for the product.

Example 2. $7a$ multiplied into $2ac$ gives $14a^2c$, and $-3b$ multiplied into $2ac$ gives $-6abc$, and these connected in order are $14a^2c-6abc$.

Example 3. In like manner a multiplied into $-b$ produces $-ab$ and 5 into $-b$ produces $-5b$, and the whole product is $-ab-5b$.

$$\begin{array}{r} 4. \\ ab+4\sqrt{x+y} \\ 3c \\ \hline 3abc+12c\sqrt{x+y} \end{array}$$

$$\begin{array}{r} 5. \\ 3x-4z \\ 2y \\ \hline 6xy-8yz \end{array}$$

$$\begin{array}{r} 6. \\ 3a^2-1 \\ -2a \\ \hline -6a^3+2a \end{array}$$

$$\begin{array}{r} 7. \\ -2x^2+x \\ -3x \\ \hline 6x^3-3x^2 \end{array}$$

$$\begin{array}{r} 8. \\ 4y+y^2 \\ 2xy \\ \hline 8xy^2+2xy^3 \end{array}$$

$$\begin{array}{r} 9. \\ 2ax-3z \\ 4b \\ \hline \end{array}$$

MULTIPLICATION.

10.

$$\begin{array}{r} 3a-4b+4 \\ -5y \\ \hline \end{array}$$

11.

$$\begin{array}{r} 6ax^2-a^2-1 \\ 3ax^2 \\ \hline \end{array}$$

12.

$$\begin{array}{r} 4+3a-x^2 \\ -2y \\ \hline \end{array}$$

CASE III.

When both factors are compound quantities.

RULE.

Multiply the first term of the multiplier into every term of the multiplicand, and place the products with their proper signs in a line below, as in case 2.

Multiply the second term of the multiplier into every term of the multiplicand, and place these products with their proper signs under the forementioned products.

Proceed in this manner with every term of the multiplier, observing to set the like quantities which occur under each other.

Add these products together, the sum will be the product required.

EXAMPLES*.

1.

$$\begin{array}{r} a+b \\ a+b \\ \hline a^2+ab \\ +ab+b^2 \\ \hline a^2+2ab+b^2 \end{array}$$

2.

$$\begin{array}{r} 2x-3z \\ 4x-5z \\ \hline 8x^2-12xz \\ -10xz+15z^2 \\ \hline 8x^2-22xz+15z^2 \end{array}$$

* The learner is cautioned to avoid an error which is frequently made in this place. He has heard that like signs produce $+$, and unlike

$$\begin{array}{r}
 3. \\
 2a + 3x \\
 a - x \\
 \hline
 2a^2 + 3ax \\
 -2ax - 3x^2 \\
 \hline
 2a^2 + ax - 3x^2
 \end{array}$$

Explanation.

Example 1. Here the multiplicand $a+b$ multiplied into a , the first term of the multiplier, by case 2, produces a^2+ab , the first line of the product; and the same, $a+b$ multiplied into b , the other term of the multiplier, produces $ab+b^2$, which is the second line; this being placed under the first, and both added together, the result is $a^2+2ab+b^2$, as in the example.

Example 2. Here $2x-3x$ multiplied into $4x$, gives the first line $8x^2-12xz$; and the same, $2x-3x$ multiplied into $-5x$ gives $-10xz+15x^2$, which is the second line; these being placed, as in the example, (viz. like quantities under each other) and added together, the product $8x^2-22xz+15x^2$ is obtained.

Example 3. $2a+3x$ multiplied into a produces $2a^2+3ax$, and the same, $2a+3x$ multiplied into $-x$, produces $-2ax-3x^2$; these added together give $2a^2+ax-3x^2$, the product required.

unlike — in multiplication; and he applies the rule, not only in multiplying, which is right, but likewise in adding up the terms of the product, which is wrong: the rule belongs to the multiplying only; the adding depends on the rules of addition.

MULTIPLICATION.

4.

$$2a - 5y$$

$$a - 2y$$

$$2a^2 - 5ay$$

$$-4ay + 10y^2$$

$$2a^2 - 9ay + 10y^2$$

5.

$$a + z$$

$$a - z$$

$$a^2 + az$$

$$-az - z^2$$

$$a^2 - z^2$$

6.

$$x^2 + 2xy + y^2$$

$$x + y$$

$$x^3 + 2x^2y + xy^2$$

$$x^2y + 2xy^2 + y^3$$

$$x^3 + 3x^2y + 3xy^2 + y^3$$

7.

$$x + 8$$

$$x - 4$$

$$x^2 + 8x$$

$$-4x - 32$$

$$x^2 + 4x - 32$$

8.

$$4x - 5a - 2b$$

$$3x - 2a + 5b$$

$$12x^2 - 15ax - 6bx$$

$$-8ax + 10a^2 + 4ab$$

$$20bx - 25ab - 10b^2$$

$$12x^2 - 23ax + 14bx + 10a^2 - 21ab - 10b^2$$

9.

$$x^3 - y^3 - z^3$$

$$x - y - z$$

$$x^4 - xy^3 - xz^3 - x^3y + y^4 + yz^3 - x^3z + y^3z + x^4$$

DIVISION.

35

Explanation.

In this example none of the terms in the product are alike, they are therefore placed in the same line, with their proper signs; and this method is to be followed in all similar cases.

EXAMPLES FOR PRACTICE.

1. Multiply $2ax$ into $4a$. *Ans.* $8a^2x$.
2. Multiply $ax-12$ into $3ay$. *Ans.* $3a^2xy-36ay$.
3. Multiply $2x-3y+z$ into $4x^2y$.
Ans. $8x^3y-12x^2y^2+4x^2yz$.
4. Multiply $5a+4b$ into $3a-2b$.
Ans. $15a^2+2ab-8b^2$.
5. Multiply a^2+ac-c^2 into $a-c$.
Ans. $a^3-2ac^2+c^3$.
6. Multiply $a+x$ into $a-z$. *Ans.* a^2-x^2 .
7. Multiply $x^2+2xy+y^2$ into $x+y$.
Ans. $x^3+3a^2y+3xy^2+y^3$.
8. Multiply x^2+xy+y^2 into $x-y$. *Ans.* x^3-y^3 .
9. Multiply $a+b-b$ into $a-b$.
Ans. a^2-ad-b^2+bd .
10. Multiply $3x^2-7xz$ into $x-8z$.
Ans. $3x^3+17x^2z-56xz^2$.
11. Multiply $b^3+b^2x+bx^2+x^3$ into $b-x$.
Ans. b^4-x^4 .

DIVISION*.

Division is the converse of multiplication; it teacheth to find how often one quantity, called the divisor, is contained in another quantity called the dividend.

* As it is evident from the nature of this rule, that the divisor, multiplied into the quotient, will produce the dividend, it follows that the signs of the divisor and quotient must be such, as will by the rule for multiplication, produce the sign of the dividend, consequently,

1. When

The dividend and divisor are called terms, and the quantity which arises from the operation, is the quotient.

General rule for the signs in all the cases of division.

When the signs of the terms are alike (viz. both +, or both —) that of the quotient will be +.

When the signs of the terms are unlike (viz. one +, and the other —) the sign of the quotient will be —.

CASE I.

When the terms are both simple quantities.

RULE I.

Place the dividend above the divisor, with a line between, like a fraction, then

If the terms consist of the same letters, without coefficients, and the dividend be the greater, subtract

1. When both terms are +, the quotient must be +; for + in the divisor, must have + in the quotient to produce + in the dividend.

2. When the terms are both —, the quotient must also be +; for — in the divisor must have + in the quotient, to produce — in the dividend.

3. When one of the terms is +, and the other —, the quotient must be —; for + in the divisor must have — in the quotient, to produce — in the dividend, and — in the divisor must have — in the quotient to produce + in the dividend. These particulars may be briefly expressed in one view, thus:

	Divisor.	Dividend.	Quotient.
1.	+	+	+
2.	—	—	+
3.	+	—	—
	—	+	—

Therefore in division, as well as in multiplication, like signs produce +, and unlike —.

DIVISION.

37

the indices of the letters in the divisor, from those of the same letters in the dividend, and place each remainder as an index, over the letter to which it belongs, in the quotient.

EXAMPLES.

1. Divide x^5 by x^2 .

$$\text{Ans. } \frac{x^5}{x^2} = x^3.$$

2. Divide $a^4 b^2$ by $a^3 b$.

$$\text{Ans. } \frac{a^4 b^2}{a^3 b} = a b.$$

Explanation.

Example 1. Here the signs of the terms being +, that of the quotient is +, and x^5 being placed above, and x^2 below the line, the index of the divisor, viz. 2, is subtracted from that of the dividend, viz. 5, and the remainder 3 placed over x , gives the quotient x^3 .

Example 2. The signs of the terms being unlike the quotient, is —, and the terms being placed as before, subtract 3 from 4, and 1 remains for the index of a , then subtract 1 from 2, and the remainder is 1 for the index of b , the quotient therefore is $-a^1 b^1$, or (because when the index is 1 it need not be written) $-a b$.

3. Divide $a^5 x^6 z^4$ by $-a^3 x^2 z^4$.

$$\text{Ans. } \frac{a^5 x^6 z^4}{-a^3 x^2 z^4} = -a^2 x^4.$$

4. Divide $ab^2 cdx$ by abx .

$$\text{Ans. } \frac{ab^2 cdx}{abx} = bcd.$$

E

5. Divide $a^4b^3c^5$ by abc^2 .

$$\text{Ans. } \frac{a^4b^3c^5}{abc^2} = a^3b^2c^3.$$

6. Divide $x^4y^3z^2$ by $-x^4y^2z$, and $-a^3x^2y^4z^3$ by a^2xyz^5 .

$$\text{Ans. } -yz \text{ and } -axy^3.$$

7. Divide $-a^4xy^2$ by $-axy$.

$$\text{Ans. } a^3y.$$

RULE 2.

If the terms have coefficients, and the dividend be the greater.

Divide the coefficient of the dividend by that of the divisor, the result is the coefficient of the quotient, to which annexing the literal part (found by rule 1) the quotient is obtained.

EXAMPLES.

1. Divide $12ax$ by $4a$. 2. Divide $24a^2bc$ by $-6ab$.

$$\text{Ans. } \frac{12ax}{4a} = 3x$$

$$\text{Ans. } \frac{24a^2bc}{-6ab} = -4ac.$$

Explanation.

Example 1. Here the signs are like, so the sign of the quotient is $+$ (understood); next, 12 divided by 4 gives 3 for the numeral part; and a being expunged by rule 1, there remains x for the literal part; the quotient is therefore $3x$.

Example 2. The signs being unlike, the quotient will be $-$; and 24 divided by 6 gives 4 for the numeral part; then expunging ab , the quotient is $-4ac$.

DIVISION.

39

3. Divide $8a^3b$ by $2a$, and $4x^2$ by $2x$.

$$\text{Ans. } \frac{8a^3b}{2a} = 4a^2b, \text{ and } \frac{4x^2}{2x} = 2x.$$

4. Divide $-4axy^2$ by axy , and $2xy$ by x .

$$\text{Ans. } \frac{-4axy^2}{axy} = -4y, \text{ and } \frac{2xy}{x} = 2y.$$

5. Divide $8axy$ by $-4ax$, and $-ab$ by a .

$$\text{Ans. } \frac{8axy}{-4ax} = -2y, \text{ and } \frac{-ab}{a} = -b.$$

6. Divide $-12x^3y$ by $-2x$, and xy by y .

$$\text{Ans. } \frac{-12x^3y}{-2x} = 6x^2y, \text{ and } \frac{xy}{y} = x.$$

7. Divide $24ab$ by $-6a$, and $-36a^2c$ by $4a$.

$$\text{Ans. } -4b, \text{ and } -9ac.$$

8. Divide $-15x^3yz$ by $-3xy$, and $21ax^2$ by $7x$.

$$\text{Ans. } 5xz, \text{ and } 3ax.$$

9. Divide $12abc$ by $-a$.

$$\text{Ans. } -12bc.$$

RULE 3.

If the dividend be greater than the divisor.

Divide the coefficients of both terms by the greatest number that will divide each without a remainder, and place each quotient opposite the term from which it is derived.

Expunge the letters common to both terms, and annex the remaining letters each to its respective coefficient in the quotient.

DIVISION.

EXAMPLES.

1. Divide
- $24x^2yz$
- by
- $28xy^2z^2$
- .

$$\text{Ans. } \frac{24x^2yz}{28xy^2z^2} = \frac{6x}{7yz}.$$

2. Divide
- $8axy$
- by
- $-12az$
- .

$$\text{Ans. } \frac{8axy}{-12az} = -\frac{2xy}{3z}.$$

Explanation.

Example 1. Here I find by trials that 4 is the greatest number that will divide 24 and 28 without leaving a remainder; I divide first the 24 by 4, and place the quotient 6 opposite; then I divide 28 by 4, and place its quotient 7 opposite; next, by expunging xyz , common to each term, there remains x in the dividend and yz in the divisor, these joined to their corresponding coefficients, the quotient is $\frac{6x}{7yz}$.

Example 2. In this example I divide 8 and 12 each by 4, which is the greatest number that will divide them both, and the quotients are 2 and 3 respectively; I next expunge a , which is common to both, and xy remains in the dividend, and z in the divisor; these annexed to their respective numbers found above give $-\frac{2xy}{3z}$.

3. Divide
- $3abc$
- by
- $15acx$
- .

$$\text{Ans. } \frac{3abc}{15acx} = \frac{b}{5x}.$$

DIVISION.

41

4. Divide $4axy$ by $-6ax^2z$.

$$\text{Ans. } \frac{4axy}{-6ax^2z} = -\frac{2y}{3xz}.$$

5. Divide $7axy$ by $42xyz$, and $-4abc$ by $-5abx$.

$$\text{Ans. } \frac{7axy}{42xyz} = \frac{a}{6z}, \text{ and } \frac{-4abc}{-5abx} = \frac{4c}{5x}.$$

6. Divide $-8abc^2$ by $-14a^2bc$, and $7xyz$ by $-9axy^3$.

$$\text{Ans. } \frac{4c}{7a}, \text{ and } -\frac{7z}{9y^2}.$$

7. Divide axz by $-3xyz$, and $-12x^2y$ by $-18xy^2$.

$$\text{Ans. } -\frac{a}{3y}, \text{ and } \frac{2x}{3y}.$$

8. Divide $6a^2x$ by $8axyz$, and $-15ax^3$ by $21xz^2$.

$$\text{Ans. } \frac{3a}{4yz}, \text{ and } -\frac{5ax}{7z}.$$

9. Divide a by a , abc by xyz , and $3xy$ by $4xy$.

$$\text{Ans. } 1, \frac{abc}{xyz}, \text{ and } \frac{3}{4}.$$

CASE II.

When the dividend is a compound quantity.

RULE.

Divide every term in the dividend by the divisor, as in case I.

E 3

DIVISION.

EXAMPLES.

1. Divide
- $4xy+6x^2$
- by
- $2x$
- .

$$\text{Ans. } \frac{4xy+6x^2}{2x} = 2y+3x.$$

2. Divide
- $abc-acd$
- by
- ac
- .

$$\text{Ans. } \frac{abc-acd}{ac} = b-d.$$

Explanation.

Example 1. Here dividing $4xy$ by $2x$, the quotient is $2y$; also, $6x^2$ by $2x$, the quotient is $3x$; these being connected, the quotient is $2y+3x$.

Example 2. abc divided by ac , gives b ; and $-acd$ divided by ac , gives $-d$; therefore $b-d$ is the quotient required.

3. Divide
- $12ax-8ab$
- by
- $4a$
- .

$$\text{Ans. } \frac{12ax-8ab}{4a} = 3x-2b.$$

4. Divide
- $ax+x^2$
- by
- $2x$
- .

$$\text{Ans. } \frac{ax+x^2}{2x} = \frac{a}{2} + \frac{x}{2} \text{ or } \frac{a+x}{2}.$$

5. Divide
- $ax-xy+x^2$
- by
- $4x$
- .

$$\text{Ans. } \frac{ax-xy+x^2}{4x} = \frac{a}{4} - \frac{y}{4} + \frac{x}{4} \text{ or } \frac{a-y+x}{4}.$$

6. Divide $3abx^2 - 4bx - 8$ by $-2ab$.

$$\text{Ans. } \frac{3abx^2 - 4bx - 8}{-2ab} = -\frac{3x^2}{2} + \frac{2x}{a} + \frac{4}{ab}.$$

7. Divide $10xz + 15xy$ by $5x$.

$$\text{Ans. } 2z + 3y.$$

8. Divide $15ax - 27x$ by $3x$.

$$\text{Ans. } 5a - 9.$$

9. Divide $18x^2 - 9x$ by $-9x$.

$$\text{Ans. } -2x + 1.$$

10. Divide $abc - bcd - bcx$ by bc .

$$\text{Ans. } a - d - x.$$

11. Divide $a^2 - 2ax + x^2$ by $-a$.

$$\text{Ans. } -a + 2x - \frac{x^2}{a}.$$

12. Divide $14a^2 - 7ab + 21ax - 28$ by $7a$.

$$\text{Ans. } 2a - b + 3x - \frac{4}{a}.$$

CASE III.

When the divisor and dividend are both compound quantities.

RULE*.

Place the divisor on the left of the dividend, as in, what is called, long division in arithmetic; observing to range the terms of both in the same order, and so that the highest power of some letter may stand first, the next highest second, and so on.

Divide the first term of the dividend by the first term of the divisor, by the former cases, and place the result in the quotient.

* This rule differs nothing from the foregoing rules in division, except that the work of multiplication, subtraction, and bringing down terms is, in those rules, performed in the mind only, whereas here the whole operation (for the sake of convenience) is actually exhibited.

Multiply the whole divisor into this result, and place the product under the dividend, observing to put like quantities under like.

Subtract this product from that part of the dividend under which it stands, and to the remainder bring down one (or more if necessary) of those terms in the dividend which have not been used.

Of this last line, divide the first term by the first term of the divisor—place the result with its proper sign in the quotient—multiply the whole divisor into this result—place the product under the forementioned last line—subtract—bring down, &c. as before, till all the terms in the dividend are brought down, and the work is finished.

When any quantity remains place it over the divisor like a fraction, and annex the whole with its proper sign to the quotient.

Methods of Proof.

Multiply the quotient into the divisor, and add the remainder (if any) to the product, the result will be the same as the dividend when the work is right.

Or,

Add the remainder and lower lines in each compartment together, and the sum will be like the dividend.

DIVISION.

45

EXAMPLES.

1. Divide $x^2 + 2xz + z^2$ by $x + z$,

The work will stand thus—

Divisor. Dividend. Quotient.

$x + z$ $x^2 + 2xz + z^2$ $(x + z$

$*x^2 + xz \dots = \text{the divisor} \times x, \text{ viz. } \overline{x + z} \times x.$

$xz + z^2 = \text{the remainder with } z^2 \text{ br'd. down.}$

$*xz + z^2 = \overline{x + z} \times z.$

$x^2 + 2xz + z^2 \dots \text{proof by addition.}$

Proof by multiplication.

$x + z \dots \text{divisor.}$

$x + z \dots \text{quotient.}$

$x^2 + xz$

$xz + z^2$

$x^2 + 2xz + z^2 \text{ dividend.}$

Explanation.

In this example the terms being placed according to the rule, I first divide x^2 by x , and place the result x for the first member of the quotient; I next multiply the divisor $x + z$ into the said result x , and place the product $x^2 + xz$ under like terms in the dividend; I then subtract the said $x^2 + xz$ from those terms, and the remainder is xz , which, by bringing down z^2 , becomes $xz + z^2$; I now divide the first term of this by the first term of the divisor, viz. xz by x and the

result is z , which multiplied into the divisor, viz. $x+z \times z$ produces $xz+z^2$, and subtracting this from the quantity which stands over it, nothing remains, and all the quantities in the dividend being used, the work is finished.

The addition proof arises by adding the lower lines marked * together, the proof by multiplication is sufficiently obvious, being the product of the divisor multiplied into the quotient.

2. Divide $a^3+3a^2x+3ax^2+x^3$ by $a+x$.

$$a+x) a^3+3a^2x+3ax^2+x^3 (a^2+2ax+x^2$$

$$*a^3 + a^2x \dots\dots\dots = a+x \times a^2$$

$$2a^2x+3ax^2 \dots = \text{there, with } 3ax \text{ br. down}$$

$$*2a^2x+3ax^2 \dots = a+x \times 2ax.$$

$$ax^2+x^3 = \text{the rem. with } x^3 \text{ br. down}$$

$$*ax^2+x^3 = a+x \times x^2.$$

$$a^3+3a^2x+3ax^2+x^3 \text{ proof by addition.}$$

Proof by multiplication.

$$a^2+2ax+x^2 \dots\dots \text{quotient.}$$

$$a+x \dots\dots\dots \text{divisor.}$$

$$a^3+2a^2x+ax^2$$

$$a^2x+2ax^2+x^3$$

$$a^3+3a^2x+3ax^2+x^3 \text{ dividend.}$$

DIVISION.

47

3. Divide $a^3 + 5a^2x + 5ax^2 + x^3$ by $a + x$.

$$\begin{array}{r} a+x \overline{) a^3 + 5a^2x + 5ax^2 + x^3} \end{array}$$

$$\begin{array}{r} 4a^2x + 5ax^2 \\ 4x^2x + 4ax^2 \end{array}$$

$$\begin{array}{r} ax^2 + x^3 \\ ax^2 + x^3 \end{array}$$

0

$$\underline{a^3 + 5ax^2 + 5ax^2 + x^3} \text{ proof by addition.}$$

4. Divide $a^2 - b^2 - 2ad + d^2$ by $a - b - d$.

$$\begin{array}{r} a-b-d \overline{) a^2 - b^2 - 2ad + d^2} \end{array}$$

$$\begin{array}{r} ab - b^2 - ad \\ ab - b^2 - bd \end{array}$$

$$\begin{array}{r} -ad + bd + d^2 \\ -ad + bd + d^2 \end{array}$$

0

$$\underline{a^2 * -b^2 - 2ad * +d^2} \text{ proof.}$$

DIVISION.

4. Divide $x^3 - 6x - 9$ by $x - 3$.

$$\begin{array}{r}
 x-3 \overline{) x^3 - 6x - 9} \quad (x^2 + 3x^2 + 3 \\
 \underline{x^3 - 3x^2} \quad \dots \text{Here } -3x^2 \text{ having no like} \\
 \phantom{x-3 \overline{) }} 3x^2 - 6x \quad \text{quantity in the dividend, I} \\
 \phantom{x-3 \overline{) }} \underline{3x^2 - 9x} \quad \text{subtract it, viz. put it down} \\
 \phantom{x-3 \overline{) }} \quad \text{with its sign changed from } - \\
 \phantom{x-3 \overline{) }} \quad \text{to } +, \text{ and bring down } -6x \\
 \phantom{x-3 \overline{) }} 3x - 9 \text{ to it.} \\
 \phantom{x-3 \overline{) }} \underline{3x - 9} \\
 \phantom{x-3 \overline{) }} 0
 \end{array}$$

$$\begin{array}{r}
 x^3 \quad * \quad -6x - 9 \text{ proof.} \\
 \hline
 \hline
 \end{array}$$

5. Divide $a^3 - 3a^2y + 3ay^2 - y^3$ by $a - y$.

$$\begin{array}{r}
 a-y \overline{) a^3 - 3a^2y + 3ay^2 - y^3} \quad (a^2 - 2ay + y^2 \\
 \underline{a^3 - a^2y} \\
 \phantom{a-y \overline{) }} -2a^2y + 3ay^2 \\
 \underline{-2a^2y + 2ay^2} \\
 \phantom{a-y \overline{) }} ay^2 - y^3 \\
 \underline{ay^2 - y^3} \\
 \phantom{a-y \overline{) }} 0
 \end{array}$$

$$\begin{array}{r}
 a^3 - 3a^2y + 3ay^2 - y^3 \text{ proof.} \\
 \hline
 \hline
 \end{array}$$

6. Divide $z^3 + 8z + 15$ by $z + 3$.Ans. $z + 5$.

DIVISION.

49

6. Divide $x^3 - 10x^2 + 33x - 36$ by $x - 4$.

$$\begin{array}{r} x-4 \overline{) x^3 - 10x^2 + 33x - 36} \end{array}$$

$$\underline{x^3 - 4x^2}$$

$$-6x^2 + 33x$$

$$\underline{-6x^2 + 24x}$$

$$9x - 36$$

$$\underline{9x - 36}$$

0

$$\underline{x^3 - 10x^2 + 33x - 36} \text{ proof.}$$

7. Divide $6a^4 - 96$ by $3a - 6$.

$$\begin{array}{r} 3a-6 \overline{) 6a^4 - 96} \end{array}$$

$$\underline{6a^4 - 12a^3}$$

$$12a^3$$

$$\underline{12a^3 - 24a^2}$$

$$24a^2$$

$$\underline{24a^2 - 48a}$$

$$48a - 96$$

$$\underline{48a - 96}$$

$$6a^4$$

$$*$$

$$*$$

$$*$$

$$*$$

$$\underline{-96} \text{ proof.}$$

F

Here -96 is not brought down till it is wanted, viz. in the last line.

DIVISION.

8. Divide $a^3 - 3a^2x + 4ax^2 - 2x^3$ by $a^2 - 2ax + x^2$

$$(a^2 - 2ax + x^2) a^3 - 3a^2x + 4ax^2 - 2x^3 (a - x + \frac{ax - x^3}{a^2 - 2ax + x^2})$$

$$\begin{array}{r} - a^2x + 3ax^2 - 2x^3 \\ - a^2x + 2ax^2 - x^3 \end{array}$$

$ax^2 - x^3$ remainder.

$$a^3 - 3a^2x + 4ax^2 - 2x^3 \text{ proof.}$$

In this example the remainder is placed above the divisor, and the whole annexed to the quotient.

9. Divide $x^2 - 2xy + y^2$ by $x - y$. *Ans.* $x - y$.

10. Divide $a^2 - b^2$ by $a - b$. *Ans.* $a + b$.

11. Divide $x^3 - 3x^2z + 3xz^2 - z^3$ by $x - z$. *Ans.* $x^2 - 2xz + z^2$.

12. Divide $x^4 - y^4$ by $x - y$. *Ans.* $x^3 + x^2y + xy^2 + y^3$.

13. Divide $x^3 - 9x^2 + 27x - 27$ by $x^2 - 6x + 9$. *Ans.* $x - 3$.

* 14. Divide $a^4 - 2a^2z$ by $a + z$. *Ans.* $a^3 - a^2z + az^2 - z^3 - \frac{z^4}{a - z}$.

* *Remarks on the preceding Rules.*

1. The sum of quantities is found by addition, and their difference by subtraction.
2. Quantities which are alike can be actually added to, or subtracted from each other, but those which are unlike cannot.
3. The sum of unlike quantities is denoted by writing the quantities one after another with their proper signs; it can be denoted no other way.

DIVISION

57

Promiscuous examples to exercise the learner in the preceding rules.

1. Required the sum, difference, product, and quotient of a and b .

Ans. Sum $a+b$, diff. $a-b$, prod. ab , quot. $\frac{a}{b}$.

2. Required the sum, difference, product, and quotient of $2ax$ and $-ax$.

Ans. Sum ax , diff. $3ax$, prod. $-2a^2x^2$, quot. -2 .

3. Required the sum, difference, product, and quotient of $a+b$ and $a-b$.

Ans. Sum $2a$, diff. $2b$, prod. a^2-b^2 , quot. $1+\frac{2b}{a-b}$.

4. The difference of unlike quantities is denoted by putting down the quantities to be subtracted with their signs changed.
5. Addition and subtraction are performed by the coefficients only; these operations never alter the indices.
6. The product of quantities is found by multiplication.
7. Quantities, whether like or unlike, can be multiplied together; and the products may be exhibited different ways, thus, the product of a , b , and c , is either abc , or $a \times b \times c$, or $ab \times c$, or $a \times bc$, or $ac \times b$, and $\overline{a+c} \times b$ is $\overline{ad+cd}$, or $(a+c) \cdot d$ or $ad+c \times d$, and x^2-y^2 is also $\overline{x+y} \times \overline{x-y}$.
8. If one of the factors be 0 the product will be 0.
9. The quotient of quantities is found by division: it can be actually found, only when the dividend contains the divisor; when that is not the case, the quotient is denoted by placing the dividend above the divisor, and expunging the quantities common to both.
10. The quotient of the same quantities may be variously written or denoted; thus, $\overline{ac+b}$ divided by a , is either $\overline{ac+b} \div a$, or $\frac{ac+b}{a}$, or $c+\frac{b}{a}$, or $c+b \div a$.
11. Multiplication and division affect the coefficients; they likewise affect the indices of those letters which are found in both terms.

4. Required the sum, difference, product, and quotient of $+a$ and $-a$.

Ans. Sum 0, diff. $2a$, prod. $-a^2$, quot. -1 .

5. Required the sum, difference, product, and quotient of $3a^2$ and $-2a^2$.

Ans. Sum a^2 , diff. $5a^2$, prod. $-6a^4$, quot. $-\frac{3}{2}$.

6. Find the sum, difference, product, and quotient of $3a+1$ and $3a-1$.

Ans. Sum $6a$, diff. 2 , prod. $9a^2-1$, quot. $1+\frac{2}{3a-1}$.

7. Required the sum, difference, product, and quotient of $x^2+2xz+x^2$ and $x^2-2xz+x^2$.

Ans. Sum $2x^2+2x^2$, diff. $4xz$, prod. $x^4-2x^2x^2+x^4$,

quot. $1+\frac{4xz}{x^2-2xz+x^2}$.

8. What is the sum, difference, product, and quotient of x^2y and xy^2 ?

9. What is the sum, difference, product, and quotient of bcd and $-xyz$?

10. Required the sum, difference, product, and quotient of x^2+ax and $ax-x^2$.

11. Find the sum and product of $a^2-2ay+y^2$ and $a-y$.

12. Find the difference and quotient of $3a-4b$ and $2a+3b$.

13. Required the product and difference of $4+a$ and $2-2a$.

14. Required the sum and quotient of $12+12x$ and $1+x$.

FRACTIONS*.

Algebraic fractions are like vulgar fractions in arithmetic, and have the same names and rules; thus—

A fraction is expressed by two quantities written one over the other with a line between them, as $\frac{a}{b}$.

The quantity a above the line is called the numerator, and that below, as b , the denominator.

An integer is properly one whole thing, in opposition to a fraction, which is a part or parts; but the word is sometimes used to signify any whole number or quantity connected with a fraction.

A proper fraction is one whose denominator will not divide its numerator.

An improper fraction is one whose denominator will divide its numerator.

A mixed quantity consists of an integer and a fraction.

A complex fraction is one whose terms are composed of mixed quantities.

Thus, $\frac{a}{b}$ is, or $\frac{2}{3}$, are proper fractions.

* A fraction is properly the quotient of the numerator divided by the denominator, as many different ways therefore as this quotient can be written, so many ways can the value of the fraction be exhibited; this should be remembered in going through the following rules, as it accounts for their operations; reduction of fractions being nothing more than certain methods of changing them from one form to another without altering their value.

A perfect acquaintance with these rules is absolutely necessary, as the resolution of problems depends, in a great measure, on the doctrine of fractions.

$\frac{ab}{b}$ or $\frac{7}{3}$, improper fractions.

$a + \frac{a}{b}$ or $7\frac{1}{3}$, mixed quantities.

$$\frac{a + \frac{b}{c}}{d} \text{ or } \frac{4\frac{1}{2}}{5}$$

and

$$\frac{\frac{a}{b} + \frac{c}{d}}{e} \text{ or } \frac{4}{5\frac{1}{2}}$$

also,

$$\frac{\frac{a}{b} - \frac{c}{d}}{e} \text{ or } \frac{4\frac{1}{2}}{5\frac{1}{2}}$$

Complex fractions.

1. To reduce a mixed quantity to an improper fraction.

RULE*.

Multiply the integer into the denominator of the fraction, and to this product add the numerator, the denominator being placed under this sum, will give the improper fraction required.

* The integer being both multiplied and divided by the same quantity, (viz. the denominator of the fraction) its value is not altered; and that the fraction which denotes the quotient of the numerator, divided by the denominator, denotes the same after the operation appears by inspection; consequently the given quantity is equal to the improper fraction to which it is reduced.

FRACTIONS.

55

EXAMPLES.

1. Reduce $a + \frac{b}{c}$. . . $a - \frac{b}{c}$ and $6\frac{7}{8}$ to improper fractions.

(1). (2). (3).

$$a + \frac{b}{c} = \frac{a \times c + b}{c} = \frac{ac + b}{c}, \text{ the first fraction.}$$

$$a - \frac{b}{c} = \frac{a \times c - b}{c} = \frac{ac - b}{c}, \text{ the second fraction.}$$

$$6\frac{7}{8} = \frac{6 \times 8 + 7}{8} = \frac{55}{8}, \text{ the third fraction.}$$

Explanation.

Here, under the number (1) the given fractions are placed; under (2) the operations are denoted; and under (3) the operations are actually performed.

In the first fraction the integer a multiplied into the denominator c , with the numerator b added, is written thus, $a \times c + b$, which, by placing the denominator c under it, becomes $\frac{a \times c + b}{c}$, and by actually

multiplying a into c , it becomes $\frac{ac + b}{c}$.

In the second fraction the integer a multiplied into c produces ac , to which, adding $-b$, and placing the denominator c under the result, it becomes $\frac{ac - b}{c}$.

In like manner 6 multiplied into 8 is 48 , and 7 added is 55 , under which placing the denominator 8 , the third fraction is $\frac{55}{8}$.

2. Reduce $x + \frac{x}{y}$ and $3a - \frac{4b}{5c}$ to improper fractions.

$$x + \frac{x}{y} = \frac{x \times y + x}{y} = \frac{xy + x}{y} \text{ the first fraction.}$$

$$3a - \frac{4b}{5c} = \frac{3a \times 5c - 4b}{5c} = \frac{15ac - 4b}{5c}, \text{ the second.}$$

3. Reduce $ax - \frac{4}{5}$ and $4 + \frac{a}{b}$ to improper fractions.

$$ax - \frac{4}{5} = \frac{ax \times 5 - 4}{5} = \frac{5ax - 4}{5}, \text{ the first fraction.}$$

$$4 + \frac{a}{b} = \frac{4 \times b + a}{b} = \frac{4b + a}{b}, \text{ the second.}$$

4. Reduce $a + \frac{y}{z}$ and $4\frac{3}{5}$ to improper fractions.

$$\text{Ans. } \frac{az + y}{z} \text{ and } \frac{23}{5}$$

5. Reduce $x + \frac{a+x}{y}$ and $a - \frac{a+b}{c}$ to improper fractions.

$$\text{Ans. } \frac{xy + a + x}{y} \text{ and } \frac{ac - a + b}{c} \text{ or } \frac{ac - a - b}{c}$$

* If any difficulty occurs here, it must be remembered that the sign — belongs to, and governs the whole fraction $\frac{a+b}{c}$; and the meaning of the expression $a - \frac{a+b}{c}$ is, that the whole fraction is to be subtracted, that is annexed to a , with the signs of all its terms changed; the same is to be understood in example 7, and where-ever such expressions occur.

FRACTIONS.

57

6. Reduce $4\frac{7}{11}$, $ax+\frac{3}{5}$ and $x+y-\frac{4}{z}$ to improper fractions.

Ans. $\frac{51}{11}$, $\frac{5ax+y}{5}$ and $\frac{xz+yz-4}{z}$.

7. Reduce $3xy-\frac{1-x}{4}$ and $2ax-\frac{x+y}{z}$ to improper fractions.

Ans. $\frac{12xy-1+x}{4}$ and $\frac{2axz-x-y}{z}$.

8. Reduce $a+\frac{b-c+4}{ax}$ $x-\frac{7}{3b}$ and $17\frac{1}{2}$ to improper fractions.

2. To reduce an improper fraction to its proper terms, that is, to a whole or mixed quantity.

RULE.*

Divide the numerator by the denominator, and the quotient will be the integral part, and place the remainder (if any) over the denominator for the fractional part; this annexed by its proper sign to the integral part, will give the mixed quantity required.

EXAMPLES.

1. Reduce the improper fractions $\frac{ac+b}{c}$ $\frac{ac-b}{c}$ and $\frac{55}{8}$ to their proper terms.

* This problem being the reverse of the former, evidently requires a contrary process; the reason of which is sufficiently obvious.

$$\frac{ac+b}{c} = a + \frac{b}{c} \text{ the first.}$$

$$\frac{ac-b}{c} = a - \frac{b}{c} \text{ the second.}$$

$$\frac{55}{8} = 6\frac{7}{8} \text{ the third.}$$

Explanation.

First, ac divided by c gives a ; and because the second term b is not divisible by c , the division can only be denoted by placing it over the denominator c ; this annexed to the integer a by its proper sign $+$ gives $a + \frac{b}{c}$ the first mixed quantity.

In like manner, $ac-b$ divided by c gives $a - \frac{b}{c}$ the second.

And 55 divided by 8 gives 6 , and 7 over, that is $6\frac{7}{8}$ for the third.

2. Reduce $\frac{xy+x}{y}$ and $\frac{15ac-4b}{5c}$ to their proper terms.

$$\frac{xy+x}{y} = x + \frac{x}{y} \text{ the first.}$$

$$\frac{15ac-4b}{5c} = 3a - \frac{4b}{5c} \text{ the second.}$$

3. Reduce $\frac{5ax-4}{5}$ and $\frac{4b^2+a}{b}$ to whole or mixed quantities.

FRACTIONS.

39

$$\frac{5ax-4}{5} = ax - \frac{4}{5} \text{ the first.}$$

$$\frac{4b^2+a}{b} = 4b + \frac{a}{b} \text{ the second.}$$

4. Reduce $\frac{24ax+2y}{3a}$ and $\frac{24}{7}$ to equivalent whole or mixed quantities.

$$\text{Ans. } 8x + \frac{2y}{3a} \text{ and } 3\frac{3}{7}.$$

5. Reduce $\frac{xy+ay+x}{y}$ and $\frac{ac-a-b}{c}$ to equivalent or mixed quantities.

$$\text{Ans. } x+a+\frac{x}{y} \text{ and } a-\frac{a+b}{c}.$$

6. Reduce the improper fractions $\frac{144}{8}$, $\frac{ax-y}{x}$ and $\frac{4a+b-c}{a}$ to their proper terms.

$$\text{Ans. } 18, a-\frac{y}{x} \text{ and } 4+\frac{b-c}{a}.$$

7. Reduce $\frac{12xy-5+x}{4}$, $\frac{112}{5}$ and $\frac{11ax+y^{\frac{3}{2}}}{11a}$ to their proper terms.

$$\text{Ans. } 3xy-1-\frac{1-x}{4}, 22\frac{2}{5} \text{ and } x+\frac{y^{\frac{3}{2}}}{11a}.$$

8. Reduce $\frac{2x-y}{x}$ and $\frac{ax+a}{a}$ to their proper terms.

$$\text{Ans. } 2-\frac{y}{x} \text{ and } x+1.$$

FRACTIONS.

3. To reduce fractions to others of equal value having a common denominator.

RULE.*

Multiply each numerator into all the denominators except its own, for a new numerator, and all the denominators together for the common denominator, over which place the new numerators severally.

EXAMPLES.

1. Reduce $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{x}{z}$ to equal fractions, having a common denominator.

* In order to understand the nature of this rule, it will be necessary to shew, that if the terms of any fraction be both multiplied, or both divided by the same quantity, the result in either case is equal to the fraction proposed; let both terms of the fraction $\frac{ax}{ay}$ be multiplied into any quantity, suppose m , the product is $\frac{amx}{amy}$. Let now the terms of the same fraction, viz. $\frac{ax}{ay}$, be divided by any quantity, that will divide both, as m , the quotient will be $\frac{ax}{ay}$.

But each of these three fractions, $\frac{amx}{amy}$, $\frac{amnx}{amny}$ and $\frac{ax}{ay}$ is equal to the same quantity, viz. to $\frac{x}{y}$, (by the rules of division) they are therefore equal to one another, consequently the value of the given fraction is not altered by such multiplication or division.

This premised, the reason of the above operations will readily appear; for of each fraction, both terms being multiplied into the same quantity (viz. the product of all the other denominators) the result is equal to the said fraction, by what has been shewn.

If each of the denominators be divisible by the same quantity, it will be better to divide them by it, and make use of the quotients instead of them; by this means the fractions will be reduced to lower terms.

$$\left. \begin{array}{l} a \times d \times z = adz \\ c \times b \times z = bcx \\ x \times b \times d = bdx \end{array} \right\} \text{new numerators.}$$

$$b \times d \times z = bdz \quad \text{common denominator.}$$

$$\text{Ans. } \frac{adz}{bdz}, \frac{bcx}{bdz} \text{ and } \frac{bdx}{bdz}.$$

Explanation.

In line 1, the first numerator a is multiplied into d and z , the denominators of the other fractions. In the next line, the numerator c is multiplied into b and z , the other denominators. In the third line, the numerator x is multiplied into b and d , the other denominators. In the fourth line, all the denominators are multiplied together. Lastly, each new numerator is placed over the common denominator.

2. Reduce $\frac{a}{x}$, $\frac{b}{z}$ and $\frac{4}{5}$ to a common denominator.

$$\left. \begin{array}{l} a \times z \times 5 = 5az \\ b \times x \times 5 = 5bx \\ 4 \times x \times z = 4xz \end{array} \right\} \text{new num.}$$

$$x \times z \times 5 = 5xz \quad \text{com. denom.}$$

$$\text{Ans. } \frac{5az}{5xz}, \frac{5bx}{5xz}, \text{ and } \frac{4xz}{5xz}.$$

3. Reduce $\frac{x}{y}$, $\frac{x+z}{4}$ and $\frac{2z}{3a}$ to a common denominator.

FRACTIONS.

$$\begin{array}{l} x \times 4 \times 3a = 12ax \\ (x+z) \times y \times 3a = 3axy + 3ayz \\ 2z \times y \times 4 = 8yz \end{array} \left. \vphantom{\begin{array}{l} x \times 4 \times 3a = 12ax \\ (x+z) \times y \times 3a = 3axy + 3ayz \\ 2z \times y \times 4 = 8yz \end{array}} \right\} \text{new num.}$$

$$y \times 4 \times 3a = 12ay \quad \text{com. den.}$$

$$\text{Ans. } \frac{12ax}{12ay}, \frac{3axy + 3ayz}{12ay} \text{ and } \frac{8yz}{12ay}.$$

4. Reduce $\frac{x}{y}$ and $\frac{a}{b}$ to equal fractions, having a common denominator.

$$\text{Ans. } \frac{bx}{by} \text{ and } \frac{ay}{by}.$$

5. Reduce $\frac{3a}{7x}$ and $\frac{4}{9}$ to a common denominator.

$$\text{Ans. } \frac{27a}{63x} \text{ and } \frac{28x}{63x}.$$

6. Reduce $\frac{a}{2x}$, $\frac{4}{y}$, and $\frac{x}{z}$ to a common denominator.

$$\text{Ans. } \frac{ayz}{2xyz}, \frac{8xz}{2xyz}, \text{ and } \frac{2x^2z}{2xyz}.$$

7. Reduce $\frac{x+y}{z}$ and $\frac{a-b}{c}$ to a common denominator.

$$\text{Ans. } \frac{cx+cy}{cz} \text{ and } \frac{ax-bz}{cz}.$$

8. Reduce $\frac{x}{a+2}$ and $\frac{3ax}{2y^2}$ to a common denominator.

$$\text{Ans. } \frac{2ay^2}{2ay^2+4y^2} \text{ and } \frac{3a^2x+6ax}{2ay^2+4y^2}.$$

FRACTIONS.

63.

9. Reduce $\frac{a}{2x}$, $\frac{x}{y}$, and $\frac{x-8}{x^2}$, to a common denominator.

$$\text{Ans. } \frac{xyz^2}{2xyz^2}, \frac{2x^2x^2}{2xyz^2}, \text{ and } \frac{2x^2y-16xy}{2xyz^2}.$$

*10. Reduce $x + \frac{a}{b}$ and $ay - \frac{4}{7}$ to a common denominator.

$$\text{Ans. } \frac{7bx+7a}{7b} \text{ and } \frac{7aby-4b}{7b}.$$

4. To reduce an integer to a fraction, having a given denominator.

RULE†.

Multiply the integer into the given denominator, and under the product place the said denominator.

EXAMPLES.

1. Reduce a and $2bx$ to equivalent fractions, having y for a denominator.

$$\text{Ans. } a = \frac{a \times y}{y} = \frac{ay}{y} \text{ and } 2bx = \frac{2bx \times y}{y} = \frac{2bxy}{y}.$$

Here the integer a is both multiplied and divided by y , as is also $2bx$.

* Those mixed quantities must be reduced to improper fractions first, and then to a common denominator.

† In this problem the integer being equally multiplied and divided, its value is not altered, the resulting fraction is therefore equal to the given integer.

2. Reduce axy and $x+y$ to fractions having $4z$ for a denominator.

$$\text{Ans. } axy = \frac{axy \times 4z}{4z} = \frac{4axyz}{4z}, \text{ the first.}$$

$$\text{and } x+y = \frac{x+y \times 4z}{4z} = \frac{4xz+4yz}{4z}, \text{ the second.}$$

3. Reduce $a-2$ and $3ay$ to fractions, having a^2x for a denominator.

$$\text{Ans. } \frac{a^3x-2a^2x}{a^2x} \text{ and } \frac{3a^3xy}{a^2x}.$$

4. Reduce ab and $4c$ to fractions, whose denominators are $-2x$.

$$\text{Ans. } -\frac{2abx}{2x} \text{ and } -\frac{8cx}{2x}.$$

5. Reduce x^2+y and $4-z$ to fractions, whose denominators are $a+b$.

$$\text{Ans. } \frac{ax^2+ay+bx^2+by}{a+b} \text{ and } \frac{4a-ax+4b-bz}{a+b}.$$

5. To reduce a fraction to its lowest terms.

RULE I*.

When one of the terms of the fraction is a simple quantity.

* The reasons on which this rule is founded are contained in the foregoing notes.

The following observations will be found useful in reducing the numeral part.

1. All numbers ending in an even number, or a cipher, are divisible by 2.

2. Numbers

FRACTIONS.

65

Expunge the letters that are common to all the quantities, and divide the coefficients by the greatest number that will divide them without a remainder, as in division, the result annexed to the remaining letters is the fraction in its lowest terms.

EXAMPLES.

1. Reduce $\frac{4ab}{8ac}$ to its lowest terms.

$$\text{Ans. } \frac{4ab}{8ac} = \frac{b}{2c}$$

Explanation.

Here it appears by inspection, that 4 is the greatest number that will divide the coefficients. Dividing therefore by 4, and expunging a , we have the quotient

$$\frac{b}{2c}$$

2. Numbers ending in 5 or 0 are divisible by 5.
3. Numbers ending in 0 are divisible by 10; ending in 00 by 100; in 000 by 1000, &c.
4. If the two right-hand figures of any number are divisible by 4, the whole is divisible by 4.
5. If the three right-hand figures are divisible by 8, the whole is divisible by 8.
6. In any number if the sum of its digits be divisible by 3 or 9, the number is divisible by 3 or 9.
7. If the right-hand digit be even, and the sum of all the digits be divisible by 6, the whole will be divisible by 6.
8. A number is divisible by 11 when the sum of the 1st, 3d, 5th, &c. digits, is equal to the sum of the 2d, 4th 6th, &c
9. A number is not divisible by another, when it is not divisible by the aliquot parts of the latter.

2. Reduce $\frac{5xy}{20xz}$ and $\frac{4ab}{5axy}$ to their least terms.

$$\text{Ans. } \frac{5xy}{20xz} = \frac{y}{4x}, \text{ and } \frac{4ab}{5axy} = \frac{4b}{5xy}.$$

3. Reduce $\frac{9x}{12y}$, $\frac{4y}{5y}$, and $\frac{abc}{axy}$ to their lowest terms.

$$\text{Ans. } \frac{3x}{4y}, \frac{4}{5}, \text{ and } \frac{bc}{xy}.$$

4. Reduce $\frac{12xy}{28ax}$, $\frac{3a}{4a^2}$, $\frac{7x}{21y}$, and $\frac{a^2bc}{ab^2c}$ to their lowest terms.

$$\text{Ans. } \frac{3y}{7a}, \frac{3}{4a}, \frac{x}{3y}, \text{ and } \frac{a}{b}.$$

RULE 2*.

When the terms of the fraction are compound quantities, and admit of no simple divisor.

* The truth of this rule may be shewn from example 1; for since $b-y$ measures b^2-y^2 , it will likewise measure $b \times (b^2-y^2) + y^2 \times (b-y)$.

But this quantity is $=b^3-y^3$, as appears by actually multiplying and adding according to the import of the signs. Wherefore, $b-y$ measures both terms of the given fraction.

It is also the greatest common measure; for if not, let there be a greater; then since this greater measures b^2-y^2 and b^3-y^3 , it must measure the remainder by^2-y^3 (for $b \times b^2-y^2 +$ the remainder $by^2-y^3 = b^3-y^3$).

But all the possible compound divisors (greater than $b-y$) of by^2-y^3 are itself, and $by-y^2$, neither of which will divide b^2-y^2 , con-

Range the quantities according to the dimensions of some letter, as in division.

Divide the greater term by the less, and this divisor by the remainder following, and so on dividing every divisor by its particular remainder, till nothing remains—the last divisor is the greatest common measure.

Divide both terms of the given fraction by the greatest common measure, and the quotients will be its lowest terms.

Note. In finding the common measure, if any quantity is common to each of the terms of a divisor, that quantity must be thrown out, and the remaining part of the divisor used, instead of the whole.

EXAMPLES.

1. Reduce $\frac{b^2-y^2}{b^3-y^3}$ to its lowest terms.

$$\frac{b^2-y^2)b^3-y^3(b}{b^3-by^2}$$

by^2-y^3 , here y^2 is common to both terms, therefore,

consequently the greater both does and does not measure b^2-y^2 , which is absurd. Wherefore $b-y$ is the greatest common measure; in like manner the demonstration may be applied to any other example.

This rule may be omitted for the present if the learner finds it too difficult to be understood.

by throwing out y^2 , we have $(b-y)b^2-y^2(b+y)$

$$\frac{b^2-by}{b^2-by}$$

$$\frac{by-y^2}{by-y^2}$$

$$\frac{by-y^2}{by-y^2}$$

Therefore, $b-y$ is the greatest common measure; now divide both terms of the fraction by it.

$$\frac{(b-y)b^2-y^2(b+y)}{b^2-by} \text{ numerator.}$$

$$\frac{by-y^2}{by-y^2}$$

$$\frac{by-y^2}{by-y^2}$$

$$\frac{(b-y)b^3-y^3(b^2+by+y^2)}{b^3-b^2y} \text{ denominator.}$$

$$\frac{b^2y-}{b^2y-by^2}$$

$$\frac{b^2y-by^2}{b^2y-by^2}$$

$$\frac{by^2-y^3}{by^2-y^3}$$

$$\frac{by^2-y^3}{by^2-y^3}$$

Consequently $\frac{b^2-y^2}{b^3-y^3} = \frac{b+y}{b^2+by+y^2}$ the lowest terms required.

2. Reduce $\frac{x^3 - b^2x}{x^2 + 2bx + b^2}$ to its lowest terms.

$$\begin{array}{r} x^2 + 2bx + b^2 \overline{) x^3 - b^2x} \end{array}$$

$-2bx^2 - 2b^2x$, now, dividing this remainder by $-2bx$, the quotient is $-x - b$; and dividing the divisor $x^2 + 2bx + b^2$ by this, we have

$$\begin{array}{r} x + b \overline{) x^2 + 2bx + b^2} \\ \underline{x^2 + bx} \\ bx + b^2 \\ \underline{bx + b^2} \\ 0 \end{array}$$

Therefore, $x + b$ is the greatest common measure, and dividing both terms of the fraction by it—we have

$$\begin{array}{r} x + b \overline{) x^3 - b^2x} \\ \underline{x^3 + bx^2} \\ -bx^2 - b^2x \\ \underline{-bx^2 - b^2x} \\ 0 \end{array}$$

$$\begin{array}{r} x + b \overline{) x^2 + 2bx + b^2} \\ \underline{x^2 + bx} \\ bx + b^2 \\ \underline{bx + b^2} \\ 0 \end{array}$$

Therefore, $\frac{x^2 - bx}{x + b}$ is the fraction in its lowest terms.

3. Reduce $\frac{a^5 - x^4}{a^5 - x^2 a^3}$ to its lowest terms.

$$\begin{array}{l} \text{expunging } a^3 \left. \begin{array}{l} a^5 - x^2 a^3 \\ \text{or} \\ a^2 - x^2 \end{array} \right\} \begin{array}{l} a^2 - x^2 (a^2 + x^2) \\ a^2 - a^2 x^2 \\ \hline a^2 x^2 - x^4 \\ a^2 x^2 - x^4 \\ \hline \end{array} \end{array}$$

Com. meas. $a^2 - x^2$.

Therefore, $\frac{a^5 - x^4}{a^5 - x^2 a^3} = \frac{a^2 + x^2}{a^3}$ the answer.

4. Reduce $\frac{ax + x^2}{ca + a^2 x}$ to its lowest terms.

$$\text{Ans. } \frac{x}{a}.$$

5. Reduce $\frac{a^2 - 1}{ab + b}$ to its lowest terms.

$$\text{Ans. } \frac{a - 1}{b}.$$

6. Reduce $\frac{3ax^2 - 6ax}{9a^2 x + 12a^2 x}$ to its lowest terms.

$$\text{Ans. } \frac{x^2 - 2xz}{3ax - 4ax}.$$

7. Reduce $\frac{x+a}{x^2-a^2x+ax-a^2}$ to its lowest terms.

$$\text{Ans. } \frac{1}{x-a^2}$$

8. Reduce $\frac{ab-a}{b^2-2b+1}$ to its lowest terms.

$$\text{Ans. } \frac{a}{b-1}$$

9. Reduce $\frac{x^3-y^3}{x^4-y^4}$ to its lowest terms.

$$\text{Ans. } \frac{1}{x^2+y^2}$$

10. Reduce $\frac{a+b}{a^2+2ab+b^2}$ to its lowest terms.

$$\text{Ans. } \frac{1}{a+b}$$

6. To reduce a complex fraction to a simple one.

RULE 1*.

When one of the terms is a mixed quantity.

Multiply the integral part into the denominator of the mixed term, and add its numerator to the product for one term.

Multiply the other term (viz. the unmixed part) of the fraction by the said denominator, for the other term of the fraction required.

* Equal multiplication of the terms being all that takes place, the reason of the rule is obvious.

FRACTIONS.

EXAMPLES.

1. Reduce $a + \frac{b}{c}$ to a simple fraction.

$$\text{Ans. } \frac{a + \frac{b}{c}}{d} = \frac{a \times c + b}{d \times c} = \frac{ac + b}{dc} \text{ the fraction required.}$$

2. Reduce $a - \frac{b}{c}$ and $\frac{a - \frac{x}{y}}{ax}$ to simple fractions.

$$\text{Ans. } \frac{a - \frac{b}{c}}{c} = \frac{a \times c - b}{a \times c - b} = \frac{ac - b}{ac - b}.$$

$$\text{and } \frac{a - \frac{x}{y}}{ax} = \frac{a \times y - x}{ax \times y} = \frac{ay - x}{axy}$$

3. Reduce $\frac{4\frac{1}{3}}{8}$ and $\frac{\frac{x}{ay + \frac{y-z}{5}}}{5}$ to simple fractions.

$$\text{Ans. } \frac{4\frac{1}{3}}{8} = \frac{4 \times 3 + 1}{8 \times 3} = \frac{13}{24}.$$

$$\text{and } \frac{\frac{x}{ay + \frac{y-z}{5}}}{5} = \frac{x \times 5}{ay \times 5 + y - z} = \frac{5x}{5ay + y - z}$$

4. Reduce $x + \frac{a}{2b}$ to a simple fraction.

Ans. $\frac{xz+a}{2bz}$.

5. Reduce $\frac{3}{a-\frac{4}{5}}$ and $\frac{4+\frac{x}{a}}{3y^2}$ to simple fractions.

Ans. $\frac{15}{5a-4}$ and $\frac{4a+x}{3ay^2}$.

6. Reduce $\frac{x-\frac{x}{4}}{xy}$ and $\frac{2xy}{3a-\frac{y-1}{y}}$ to simple fractions.

Ans. $\frac{4x-x}{4xy}$, or $\frac{3}{4y}$, and $\frac{2xy^2}{3ay-y+1}$.

RULE 2*.

When both terms of the fraction are mixed quantities, proceed with both terms as you did with the mixed term in rule 1; then multiply the result in each into the denominator of the other term, and the products will be the terms of the fraction required.

* That part of the above rule which depends on rule 1, being understood, the multiplication cross ways by the denominators of the terms remains to be accounted for—in order to which it must be observed, that taking away the denominator of the upper term is, in effect, multiplying that term by it; and the same may be said of the lower term: but if one term of a fraction be multiplied into any

FRACTIONS.

EXAMPLES.

1. Reduce $\frac{a + \frac{x}{y}}{x + \frac{y}{z}}$ to a simple fraction.

$$\text{Ans. } \frac{a + \frac{x}{y}}{x + \frac{y}{z}} = \frac{(a \times y + x) \times z}{(x \times z + y) \times y} = \frac{(ay + x) \times z}{(xz + y) \times y} = \frac{ayz + xz}{xyz + y^2}.$$

Explanation.

Here the integer a is multiplied into the denominator y ; and the numerator x is added to the product, and this sum is multiplied into the denominator in the other term z , which product $ayz + xz$ is the numerator of the required fraction.

In like manner the integer x is multiplied into z , and y added to the product, and this sum is multiplied into y , the denominator in the other term, this produces the denominator $xyz + y^2$.

2. Reduce $\frac{x + \frac{x}{y}}{z - \frac{y}{c}}$ to a simple fraction.

$$\text{Ans. } \frac{x + \frac{x}{y}}{z - \frac{y}{c}} = \frac{(x \times y + x) \times c}{(z \times c - y) \times y} = \frac{(xy + x) \times c}{(cz - y) \times y} = \frac{cxy + cx}{cyz - y^2}.$$

any quantity, the other term must likewise be multiplied into it, to make the result equal to the given fraction; wherefore, the upper term must be multiplied into the denominator of the lower, and the lower term into the denominator of the upper, which is the rule.

If the learner finds this rule difficult to understand, he may omit it for the present.

3. Reduce $4 + \frac{x}{x - \frac{5}{6}}$ to a simple fraction.

$$\text{Ans. } \frac{4 + \frac{x}{x - \frac{5}{6}}}{x - \frac{5}{6}} = \frac{(4 \times x + x) \times 6}{(x \times 6 - 5) \times x} = \frac{(4x + x) \times 6}{(6x - 5) \times x} = \frac{24x + 6x}{6xz - 5x}$$

4. Reduce $\frac{xy - \frac{x}{y}}{a + \frac{bx}{c}}$ to a simple fraction.

$$\text{Ans. } \frac{cxy^2 - cx}{acy + bxy}$$

5. Reduce $\frac{3 + \frac{4}{x}}{4 - \frac{5}{y}}$ and $\frac{a^2 - \frac{1}{3}}{b + \frac{1}{3}}$ to simple fractions.

$$\text{Ans. } \frac{3xy + 4y}{4xy - 5x} \text{ and } \frac{6a^2 - 3}{6b + 2}$$

6. Reduce $\frac{x - \frac{ay}{b}}{y + \frac{ax}{c}}$ to a simple fraction.

$$\text{Ans. } \frac{bcx - acy}{bcy + abx}$$

7. To add fractional quantities together.

RULE*.

Reduce the fractions to a common denominator, (if they are not so already) then add all the new numerators together for a numerator, under which place the common denominator.

If this fraction be an improper one, reduce it to its proper terms.

EXAMPLES.

1. Add $\frac{a}{5}$ and $\frac{a}{6}$ together.

$$\begin{array}{l} a \times 6 = 6a \\ a \times 5 = 5a \end{array} \left. \vphantom{\begin{array}{l} a \times 6 = 6a \\ a \times 5 = 5a \end{array}} \right\} \text{new numerators.}$$

$11a = \text{their sum.}$

$5 \times 6 = 30$, the com. denom.

Therefore $\frac{11a}{30}$ is the sum required.

Explanation.

Here the given fractions being reduced to a common denominator, the new numerators are $6a$ and

* To account for this rule it is necessary to observe, that fractions with different denominators are unlike, consequently cannot be added or subtracted—but when they are reduced to a common denominator, or made parts of the same whole, their sum or difference may be truly found by adding or subtracting the numerators, seeing the numerators in this case express the values of the fractions; wherefore the rules for addition and subtraction are manifest.

FRACTIONS.

77

5a, which added together give 11a; this placed over the common denominator 30 gives $\frac{11a}{30}$ the answer.

2. Add $a + \frac{b}{2}$, $x - \frac{b}{3}$, and $2a + \frac{3b}{4}$ together.

$$\left. \begin{array}{l} b \times 3 \times 4 = 12b \\ -b \times 2 \times 4 = -8b \\ 3b \times 2 \times 3 = 18b \end{array} \right\} \text{new num.}$$

$$30b - 8b = 22b \text{ their sum.}$$

$$2 \times 3 \times 4 = 24 \text{ com. denom.}$$

Therefore $\frac{22b}{24} = \frac{11b}{12}$ is the sum of the fractions, to which annexing the sum of the integers a, x, and 2a, which is 3a+x, the answer is $3a+x+\frac{11b}{12}$.

3. Add $\frac{a}{b}$, $\frac{c}{d}$, $\frac{x}{y}$, and $\frac{4}{7}$ together.

$$\left. \begin{array}{l} a \times d \times y \times 7 = 7ady \\ c \times b \times y \times 7 = 7bcy \\ x \times b \times d \times 7 = 7bdx \\ 4 \times b \times d \times y = 4bdy \end{array} \right\} \text{new num.}$$

$$b \times d \times y \times 7 = 7bdy \text{ com. denom.}$$

$$\text{Their sum } \frac{7ady + 7bcy + 7bdx + 4bdy}{7bdy}$$

4. Add $\frac{3a}{4}$ and $\frac{5a}{7}$ together.

$$\text{Ans. } \frac{41a}{28}$$

5. Add $\frac{3a}{2b}$ and $\frac{a}{5}$ together. *Ans.* $\frac{15a+2ab}{10b}$.

6. Add $\frac{x}{2}$, $\frac{x}{3}$, and $\frac{x}{4}$ together.
Ans. $\frac{13x}{12} = x + \frac{x}{12}$.

7. Add $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{x}{z}$ together.
Ans. $\frac{adx+bcz+bdx}{bdx}$.

8. Required the sum of $4 + \frac{x}{y}$ and $5 - \frac{a}{b}$.
Ans. $9 + \frac{bx-ay}{by}$.

9. Find the sum of $4a + \frac{a-4}{b}$ and $5a + \frac{2a-3}{c}$.
Ans. $9a + \frac{ac-4c+2ab-3b}{bc}$.

10. Add $\frac{a+4}{5}$ and $\frac{3a}{5}$ together. *Ans.* $\frac{4a+4}{5}$.

11. Required the sum of $3a$, $\frac{4x^2}{5a}$, and $3 + \frac{x}{4}$.
Ans. $3a + 3 + \frac{16x^2+5ax}{20a}$.

FRACTIONS.

79

12. Add $4x + \frac{3x}{7}$ and $x - \frac{4x}{5}$ together.

Ans. $5x - \frac{13x}{35}$.

13. Find the sum of $\frac{x}{x+1}$ and $\frac{x}{x-1}$.

Ans. $\frac{2x^2}{x^2+1}$.

8. To subtract one fractional quantity from another.

RULE.

Reduce the fractions to a common denominator as before.

Subtract one new numerator from the other, and under the difference place the common denominator; this fraction will be the difference required.

EXAMPLES.

1. From $\frac{a}{2}$ take $\frac{2a}{9}$.

$$\begin{array}{l} a \times 9 = 9a \\ 2a \times 2 = 4a \end{array} \left. \vphantom{\begin{array}{l} a \times 9 = 9a \\ 2a \times 2 = 4a \end{array}} \right\} \text{new numerators.}$$

$5a$ their difference.

$2 \times 9 = 18$ common denominator.

Ans. $\frac{5a}{18}$ the difference required.

This example is done like those in prob. 7, except that here you subtract the numerators instead of adding.

FRACTIONS.

2. Required the difference of $\frac{x}{a}$ and $\frac{y}{z}$.

$$\left. \begin{array}{l} x \times z = xz \\ y \times a = ay \end{array} \right\} \text{new num.}$$

$$xz - ay \text{ their diff.}$$

$$a \times z = az \text{ com. den.}$$

$$\text{Ans. } \frac{xz - ay}{az} \text{ the required difference.}$$

3. From $\frac{x+4}{5}$ take $\frac{x-a}{y}$.

$$\left. \begin{array}{l} \overline{x+4} \times y = xy + 4y \\ x-a \times 5 = 5x - 5a \end{array} \right\} \text{new num.}$$

$$xy + 4y - 5x + 5a \text{ diff.}$$

$$5 \times y = 5y \text{ com. denom.}$$

$$\text{Ans. } \frac{xy + 4y - 5x + 5a}{5y}.$$

4. From $\frac{x}{2}$ take $\frac{x}{3}$.

$$\text{Ans. } \frac{x}{6}.$$

5. From $\frac{3y}{z}$ take $\frac{4y}{5}$.

$$\text{Ans. } \frac{15y - 4yz}{5z}.$$

6. Find the difference of $\frac{4a}{7b}$ and $\frac{3a}{8c}$.

$$\text{Ans. } \frac{32ac - 21ab}{56bc}.$$

FRACTIONS.

81

7. From $4x$ take $\frac{3x}{5}$. *Ans.* $\frac{17x}{5}$.

8. From $\frac{x}{x+y}$ take $\frac{y}{z}$. *Ans.* $\frac{xz-xy-y^2}{xz+yz}$.

9. Required the difference of $\frac{x-y}{2a}$ and $\frac{x+y}{3a}$.

Ans. $\frac{ax-5ay}{6a^2}$.

10. From $5a+\frac{x}{y}$ take $2a-\frac{x-a}{z}$.

Ans. $3a+\frac{xz-xy+ay}{yz}$.

11. From $4+\frac{x}{3}$ take $3+\frac{2x}{9}$.

Ans. $1+\frac{3x}{27}$ or $1+\frac{x}{9}$.

9. *To multiply fractions together.*

RULE I*.

Multiply the numerators together for a new numerator, and the denominators for a new denominator.
Reduce the new fraction to its lowest terms.

* The truth of this rule may be shewn thus, let $\frac{1}{2}$ be multiplied into $\frac{2}{3}$; this is evidently the same as taking $\frac{1}{2}$ of $\frac{2}{3}$, and performing the multiplication by the rule, the product is $\frac{2}{6}=\frac{1}{3}$, which is known to be the true product, for the half of two thirds is one third.

To

EXAMPLES.

1. Multiply $\frac{2x}{5}$ into $\frac{3x}{4}$.

$$\frac{2x}{5} \times \frac{3x}{4} = \frac{6x^2}{20} \text{ (reduced) } = \frac{3x^2}{10} \text{ the product required.}$$

Explanation.

First, the numerators $2x$ and $3x$ multiplied, produce $6x^2$ for the new numerator; and 5 multiplied into 4 is 20 for the denominator; the new fraction is therefore $\frac{6x^2}{20}$, which reduced to its lowest terms is $\frac{3x^2}{10}$ the answer.

2. Multiply $\frac{a}{x}$, $\frac{3x}{y}$, and $\frac{4y}{3z}$ together.

$$\text{Ans. } \frac{a}{x} \times \frac{3x}{y} \times \frac{4y}{3z} = \frac{12axy}{3xyz} = \frac{4a}{z} \text{ the product.}$$

3. Find the product of $\frac{x+y}{3a}$ and $\frac{3y}{x-y}$.

$$\text{Ans. } \frac{x+y}{3a} \times \frac{3y}{x-y} = \frac{3xy+3y^2}{3ax-3ay} = \frac{xy+y^2}{ax-ay}$$

To make this still plainer let $\frac{ab}{a}$ be multiplied into $\frac{xy}{y}$, the work will stand thus:

$$\frac{ab}{a} \times \frac{xy}{y} = \frac{abxy}{ay} = bx \text{ the product.}$$

But $\frac{ab}{a} = b$ and $\frac{xy}{y} = x$; now if instead of the fractions $\frac{ab}{a}$ and $\frac{xy}{y}$, we multiply their equals b and x together, the product will be bx , the very same as before, wherefore the rule is manifest.

4. Multiply $\frac{3x}{4}$ and $\frac{4x}{5}$ together.

$$\text{Ans. } \frac{12x^2}{20}, \text{ or } \frac{3x^2}{5}$$

5. Multiply $\frac{x}{3y}$ into $\frac{6y}{7x}$.

$$\text{Ans. } \frac{6xy}{21xy} = \frac{2}{7}$$

6. Multiply $\frac{3a}{4b}$ into $-\frac{4a}{5}$.

$$\text{Ans. } -\frac{12a^2}{20b} = -\frac{3a^2}{5b}$$

7. Required the product of $\frac{x}{a}$ into $\frac{x+y}{x+2y}$.

$$\text{Ans. } \frac{x^2+xy}{ax+2ay}$$

8. Required the product of $\frac{4ax}{y}$, $\frac{3xy}{2a}$, and $\frac{2}{x}$.

$$\text{Ans. } \frac{24ax^2y}{2axy} = 12x.$$

9. Multiply $-\frac{a}{x}$ into $-\frac{y}{z}$.

$$\text{Ans. } \frac{ay}{xz}$$

10. Multiply $-\frac{x}{y}$, $-\frac{a}{b}$, and $\frac{3}{4}$ together.

$$\text{Ans. } \frac{3ax}{4by}$$

11. Multiply $\frac{ax}{y}$ into $\frac{ax}{4}$.

$$\text{Ans. } \frac{a^2xx}{4y}$$

RULE 2*.

If the numerator of one fraction be the same with the denominator of another, they may both be left out.

EXAMPLES.

1. Multiply $\frac{4x}{3a}$ into $\frac{3a}{b}$.

Ans. $\frac{4x}{3a} \times \frac{3a}{b} = \frac{4x}{b}$ the product.

Here $3a$ is left out, being both a numerator and a denominator.

2. Multiply $\frac{x}{y}$, $\frac{y}{2a}$, and $\frac{2a}{5}$ together.

Ans. $\frac{x}{y} \times \frac{y}{2a} \times \frac{2a}{5} = \frac{x}{5}$ the product.

In this example y and $2a$ are left out.

3. Multiply $\frac{2x}{y}$, $\frac{3x}{2x}$, and $\frac{y}{4}$ together.

Here, leaving out y and $2x$, the answer is $\frac{3x}{4}$.

4. Find the product of $\frac{a}{b}$, $\frac{c}{d}$, $\frac{x}{a}$, and $\frac{b}{x}$.

Ans. $\frac{c}{d}$.

* Equal division of the terms being all that takes place in this and the following rules, the truth of each will be manifest.

FRACTIONS.

85

RULE 3.

When the numerator of one fraction and the denominator of another can be divided by any quantity, the quotients may be used instead of that numerator and denominator.

EXAMPLES.

1. Multiply $\frac{8y}{9}$ into $\frac{6x}{16}$.

Here, dividing $8y$ and 16 by 8 ; also, $6x$ and 9 by 3 , we have

$$\frac{y}{3} \times \frac{2x}{2} = \frac{2xy}{6} = \frac{xy}{3} \text{ the product.}$$

2. Multiply $\frac{3ax}{4y}$ into $\frac{y}{2ax}$.

Here the y and ax being each common to a numerator and denominator, are left out, the product is $\frac{3}{8}$.

3. Multiply $\frac{xy}{4z}$ into $\frac{4y}{3x}$. *Ans.* $\frac{y^2}{3z}$.

4. Multiply $\frac{4a}{b}$, $\frac{3a}{8c}$, $\frac{4b}{6a}$, and $\frac{11}{12}$ together.

Ans. $\frac{11a}{12c}$.

5. Multiply $\frac{3y}{4}$, $\frac{8x}{9z}$, and $\frac{11x}{12}$ together.

Ans. $\frac{11x^2y}{18z}$.

110. To multiply an integer and fraction together.

RULE*.

Divide the denominator of the fraction by the integer, when it can be done without leaving any remainder, or when it cannot, then

Multiply the numerator into the integer, the result in both cases is the same, being the product required.

EXAMPLES.

1. Multiply $\frac{x}{8}$ into 4.

Here $\frac{x}{8} \times 4 = \frac{x}{8 \div 4} = \frac{x}{2}$ the product by the first method.

And $\frac{x}{8} \times 4 = \frac{x \times 4}{8} = \frac{4x}{8} = \frac{x}{2}$ the product by the second method.

Explanation.

First. The denominator 8 is divided by the integer 4, which gives 2 for a denominator, which placed under the numerator x the answer is $\frac{x}{2}$. Next, the

*It is evident from the nature of fractions, that multiplying one term of a fraction, or dividing the other by any (the same) quantity, produces the same result, which accounts for the former part of the rule: with respect to the latter, if 1 be written under the integer it becomes a fraction, and the operation will be found exactly the same as the foregoing rule.

The first method is the most convenient when it can be applied, the second is always practicable.

numerator x is multiplied into 4, and the product $4x$ is placed over the denominator thus, $\frac{4x}{8}$, which reduced is $\frac{x}{2}$, the very same as before.

2. Multiply $\frac{3y}{13}$ into $2x$.

$$\text{Ans. } \frac{3y \times 2x}{13} = \frac{6xy}{13} \text{ the product.}$$

3. Multiply $\frac{ax}{8ay}$ into $4a$.

$$\text{Ans. } \frac{ax}{8ay \div 4a} = \frac{ax}{2y} \text{ the product.}$$

4. Multiply $\frac{xy}{3a}$ into $3a$.

$$\text{Ans. } xy.$$

5. Multiply $\frac{3x^2}{8ax}$ into $8a$.

$$\text{Ans. } \frac{3x^2}{x}.$$

6. Multiply $3ax$ into $-\frac{ab}{12ax}$.

$$\text{Ans. } -\frac{ab}{4}.$$

7. Multiply $\frac{ay}{4x}$ into $-4x$.

$$\text{Ans. } -ay.$$

8. Multiply $\frac{x}{y}$ into $3x$.

$$\text{Ans. } \frac{3x^2}{y}.$$

9. Multiply ax , $\frac{3y}{4}$, and $\frac{a}{2b}$ together.

$$\text{Ans. } \frac{3a^2xy}{8b}.$$

11. To divide one fraction by another.

RULE 1*.

Invert the divisor and then multiply the fractions together, the result will be the quotient required.

EXAMPLES.

Divide $\frac{a}{4}$ by $\frac{a}{3}$.

$$\frac{a}{4} \times \frac{3}{a} = \frac{3a}{4a} = \frac{3}{4} \text{ the quotient.}$$

Explanation.

Here the divisor $\frac{a}{3}$ being inverted is $\frac{3}{a}$, into which the other fraction $\frac{a}{4}$ being multiplied, the result $\frac{3a}{4a}$ reduced to $\frac{3}{4}$ is the quotient.

* The reason of this rule may be thus explained: suppose it were required to divide $\frac{a}{b}$ by $\frac{c}{d}$; now $\frac{a}{b}$ divided by c is evidently $\frac{1}{c}$ part of $\frac{a}{b}$ or $\frac{a}{cb}$. But instead of c we are required to divide by $\frac{c}{d}$ part of c only (for $\frac{c}{d} = \frac{1}{d}$ of c) if therefore we multiply the quotient $\frac{a}{cb}$ into d instead of dividing c the divisor, by it, the result will be the same, viz. $\frac{ad}{cb}$, (for in division dividing the divisor is the same as multiplying the quotient.) Therefore $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{cb}$, which is the rule.

2. Divide $\frac{3x}{4}$ by $\frac{ax}{2b}$.

$$\frac{3x}{4} \times \frac{2b}{ax} = \frac{3b}{2a} \text{ the quotient.}$$

3. Divide $\frac{a+x}{a-y}$ by $\frac{a+y}{a+2x}$.

$$\frac{a+x}{a-y} \times \frac{a+2x}{a+y} = \frac{a^2+3ax+2x^2}{a^2-y^2} \text{ the quotient.}$$

4. Divide $\frac{a}{4}$ by $\frac{3a}{5}$

$$\text{Ans. } \frac{5}{12}.$$

5. Let $\frac{3x}{4}$ be divided by $\frac{4}{5}$. $\text{Ans. } \frac{15x}{16}$ or $\frac{15x}{16}$.

6. Find the quotient of $\frac{x-1}{3}$ divided by $\frac{x+1}{4}$.

$$\text{Ans. } \frac{4x-4}{3x+3}.$$

7. Divide $\frac{a+x}{y}$ by $\frac{5a}{4b}$.

$$\text{Ans. } \frac{4ab+4bx}{5ay}.$$

8. Let $\frac{5ab}{4}$ be divided by $-\frac{4a}{y}$.

$$\text{Ans. } -\frac{5by}{16}.$$

9. Find the quotient of $\frac{a}{a+1}$ divided by $\frac{3x}{4y}$.

$$\text{Ans. } \frac{4ay}{3ax+3x}.$$

10. Divide $\frac{y}{y-1}$ by $-\frac{y}{3}$. *Ans.* $-\frac{3}{y-1}$.

11. Divide $\frac{1}{x-y}$ by $\frac{1}{x^2-y^2}$. *Ans.* $x+y$.

RULE 2*.

When the terms of the divisor will exactly divide those of the dividend, then divide numerator by numerator, and denominator by denominator, the result will be the quotient required.

EXAMPLES.

1. Divide $\frac{4ax}{9ab}$ by $\frac{2x}{3a}$.

$$\frac{4ax}{9ab} \div \frac{2x}{3a} = \frac{2a}{3b} \text{ the quotient.}$$

Explanation.

First, $4ax$ divided by $2x$ gives $2a$ the numerator; then $9ab$ divided by $3a$ gives $3b$ the denominator.

2. Divide $\frac{12x}{25y}$ by $\frac{4}{5}$.

$$\text{Ans. } \frac{12x}{25y} \div \frac{4}{5} = \frac{3x}{5y} \text{ the quotient.}$$

* The truth of this rule will be evident, as the result is the same as that of rule 1; it may however be remarked, that the former rule is always practicable, the latter not always; but where the latter can be applied, the quotient will be obtained in lower terms.

FRACTIONS.

95

3. Divide $\frac{x^2-y^2}{a^2+b^2}$ by $\frac{x-y}{a+b}$.

Ans. $\frac{x^2-y^2}{a^2-b^2} \div \frac{x-y}{a+b} = \frac{x+y}{a-b}$ the quotient.

4. Divide $\frac{8x}{9y}$ by $-\frac{2x}{3y}$. *Ans.* $-\frac{4}{3} = -1\frac{1}{3}$.

5. Divide $\frac{4ab}{15c}$ by $\frac{4ab}{5}$. *Ans.* $\frac{1}{3c}$.

6. Let $\frac{x+z}{x^2+2xy+y^2}$ be divided by $\frac{x}{x+y}$.

Ans. $\frac{x+z}{x+y}$.

12. To divide a fraction by an integer.

RULE*.

Divide the numerator by the integer when it can be done without leaving a remainder; and when it cannot, multiply the denominator into the integer, the result being in both cases the same, is the quotient required.

EXAMPLES.

1. Divide $\frac{4ax}{5y}$ by $4a$.

* The reason of this rule will appear from what has been said in the note on prob. 11.

FRACTIONS.

First way, $\frac{4ax}{5y} \div 4a = \frac{4ax \div 4a}{5y} = \frac{x}{5y}$ the quotient.

Second way, $\frac{4ax}{5y} \div 4a = \frac{4ax}{5y \times 4a} = \frac{4ax}{20ay} = \frac{x}{5y}$ the same.

Explanation.

First, The numerator $4ax$ divided by $4a$ gives x , which placed over the denominator $5y$ gives $\frac{x}{5y}$ for the quotient. In the second, the denominator $5y$ is multiplied into $4a$ the divisor, the result is $\frac{4ax}{20ay}$, which reduced is $\frac{x}{5y}$, the very same as before.

2. Divide $\frac{3xy-6x}{2b}$ by $3x$.

First way, $\frac{3xy-6x}{2b} \div 3x = \frac{(3xy-6x) \div 3x}{2b} = \frac{y-2}{2b}$, the quotient.

Second way, $\frac{3xy-6x}{2b} \div 3x = \frac{3xy-6x}{2b \times 3x} = \frac{3xy-6x}{6bx} = \frac{y-2}{2b}$, the quotient as before.

3. Divide $\frac{3ay}{4}$ by $3ay$.

Ans. $\frac{1}{4}$.

4. Divide $\frac{xy}{z}$ by $-3a$.

Ans. $-\frac{xy}{3az}$.

FRACTIONS.

93

5. Divide $\frac{abc}{d}$ by abc , and $\frac{xyz}{4}$ by $7a^2$.

$$\text{Ans. } \frac{1}{d} \text{ and } \frac{xyz}{28a^2}.$$

6. Divide $\frac{x^2 - 2xy + y^2}{a - 1}$ by $x - y$. $\text{Ans. } \frac{x - y}{a - 1}$.

7. Divide $\frac{60xy}{4z}$ by $3x$, $4y$, 5 , $6x$, and 7 .

$$\text{Ans. } \frac{5y}{z}, \frac{15x}{4z}, \frac{3xy}{z}, \frac{5y}{2z}, \text{ and } \frac{15xy}{7z}.$$

13. To divide an integer by a fraction.

RULE*.

Multiply the integer into the denominator of the fraction, for a numerator; under which product place the numerator of the fraction for a denominator.

EXAMPLES.

1. Divide $3a$ by $\frac{4x}{y}$.

$$\text{Ans. } 3a \div \frac{4x}{y} = \frac{3a \times y}{4x} = \frac{3ay}{4x} \text{ the quotient.}$$

* If 1 be written as a denominator under the integer, it becomes a fraction, which being multiplied into the divisor, inverted according to prob. 11, the operation is the same as the above rule directs.

Explanation.

The integer $3a$ is here multiplied into the denominator y , and the product $3ay$ is the numerator, under this the numerator of the divisor $4x$ is placed for a denominator, making the quotient $\frac{3ay}{4x}$.

2. Divide $3x+4y$ by $\frac{4}{5}$, and 4 by $\frac{4}{7}$.

First, $\overline{3x+4y} \div \frac{4}{5} = \frac{3x+4y \times 5}{4} = \frac{15x+20y}{4} = 3x + 5y + \frac{3x}{4}$, first quotient.

Then $4 \div \frac{4}{7} = \frac{4 \times 7}{4} = \frac{28}{4} = 7$, second quotient.

3. Divide 11 by $\frac{3}{4}$, and ab by $\frac{c}{d}$.

Ans. $14\frac{2}{3}$ and $\frac{abd}{c}$.

4. Divide $3ax$ by $\frac{2}{3}$, and $-5y$ by $\frac{4}{5}$.

Ans. $4ax + \frac{ax}{2}$ and $-6y - \frac{y}{4}$.

5. Divide $x^2-2ax+a^2$ by $\frac{1}{x-a}$.

Ans. $x^2-3x^2a+3xa^2-a^3$.

*6. Divide a by $\frac{a}{2}$, and $-a^2$ by $-\frac{a^2}{3}$.

Ans. 2 and 3.

Promiscuous examples in the preceding rules for the learner's exercise.

1. Required the sum and difference of $\frac{2x}{3a}$ and $\frac{2x}{5a}$.

Ans. Sum $\frac{16x}{15a}$, diff. $\frac{4x}{15a}$.

2. Find the product and quotient of $\frac{a}{b}$ and $\frac{c}{d}$.

Ans. Prod. $\frac{ac}{bd}$, quot. $\frac{ad}{bc}$.

3. Find the sum and product of $\frac{x}{y}$ and $\frac{a}{b}$.

Ans. Sum $\frac{bx+ay}{by}$, prod. $\frac{ax}{by}$.

* The foregoing rules are of the most extensive application; for integral quantities may be reduced to the form of fractions, and their operations performed by the very same methods.

Fractions cannot be added to or subtracted from each other till they are reduced to a common denominator.

Fractions may be multiplied or divided without any previous reduction.

The sum of two fractions is greater than either, and their difference is less than the greater.

The product of two proper fractions is less than either, and the quotient of any two fractions is greater than the dividend.

Fractions having the same denominator are to each other as their numerators; and having the same numerator they are to each other inversely as their denominators.

Hence in general fractions are to each other directly as their numerators, and inversely as their denominators.

4. Required the difference and quotient of $\frac{a}{b}$ and $\frac{4}{5}$.
Ans. Diff. $\frac{5a-4b}{5b}$, quot. $\frac{5a}{4b}$.

5. Find the sum and quotient of $\frac{x}{3y}$ and $\frac{4}{7}$.

Ans. Sum $\frac{7x+12y}{21y}$, quot. $\frac{7x}{12y}$.

6. Find the quotient and difference of $\frac{14a}{15}$ and $\frac{a}{2}$.

Ans. Quot. $1\frac{13}{15}$, diff. $\frac{13a}{30}$.

7. Which is the greater $\frac{4}{7}$ or $\frac{5}{9}$, and how much?

Ans. $\frac{4}{7}$ is the greater by $\frac{1}{63}$.

8. Which is the greater the product or the quotient of $\frac{5}{8}$ by $2\frac{1}{2}$, what are they, and what is the difference?

Ans. Prod. $\frac{25}{16}$, quot. $\frac{1}{4}$, the product is greatest

by $1\frac{5}{16}$.

9. What will arise by multiplying the sum of $\frac{x}{y}$ and $\frac{2x}{4}$ into their product.

Ans. $\frac{2x^3+x^3y}{8y^2}$.

INVOLUTION.

97

10. Add the sum, difference, product, and quotient of $\frac{2a}{3}$ and $\frac{3a}{4}$ together.

$$\text{Ans. } \frac{27a + 9a^2 + 16}{18}.$$

11. Add the sum and difference of $\frac{3ax}{4y}$ and $\frac{2ax}{5z}$ together.

$$\text{Ans. } \frac{3ax}{2y}.$$

12. From the quotient of $\frac{3x}{4b}$ by $\frac{2x}{5b}$ subtract their product, and find what will arise by adding $\frac{3x}{4}$ to the remainder.

$$\text{Ans. } \frac{75b^2 - 12x^2 + 30b^2x}{40b^2}.$$

INVOLUTION.

The power of any quantity is that which arises by multiplying that quantity continually into itself; the given quantity is called the root.

Involution teacheth to find the powers of any given quantity, viz. its square, cube, biquadrate, &c.

RULES.

1. Multiply the quantity continually into itself as many times as there are units in the index of the power less one, the last product will be the power required; or
2. Multiply the index of the given root into the index of the power, the result is the power required as before.

K

Note. The first rule is most convenient for the coefficients, and the second for the literal part, when simple quantities are to be involved.

General rule for the signs.

When the sign of the root is +, all the powers of it will be + likewise; and when the root is —, all the even powers are +, and all the odd powers —.

1. Involve a to the second, third, fourth, and fifth powers.

By rule 1.

By rule 2.

$$\begin{array}{l} a \times a \dots = a^2 \dots \text{or} \dots a^1 \times 2 = a^2 \text{ Square.} \\ a \times a \times a \dots = a^3 \dots \dots a^1 \times 3 = a^3 \text{ Cube.} \\ a \times a \times a \times a \dots = a^4 \dots \dots a^1 \times 4 = a^4 \text{ 4}^{\text{th}} \text{ power.} \\ a \times a \times a \times a \times a = a^5 \dots \dots a^1 \times 5 = a^5 \text{ 5}^{\text{th}} \text{ power.} \end{array}$$

Explanation.

Here, by rule 1, a is multiplied once into itself for the square, twice for the cube, three times for the 4th power, and four times for the 5th power; but the same end is more readily attained by rule 2, where the index of a , viz. 1 is multiplied into 2 for the index of the square, 3 for the cube, 4 for the 4th power, and 5 for the 5th power.

2. Involve $-2x$ to the fifth power.

By rule 1.

By rule 2.

$$- 2x$$

$$- 2x$$

$$+ 4x^2 \text{ Square.}$$

$$- 2x$$

For the coefficients.

$$-2 \times -2 \times -2 \times -2 \times -2 = -32$$

INVOLUTION.

99

$$\begin{array}{r} - 8x^3 \text{ Cube.} \\ - 2x \\ \hline \end{array}$$

$$\begin{array}{r} - 2x \\ \hline \end{array}$$

$$+ 16x^4 \text{ Biquadrate.}$$

$$\begin{array}{r} - 2x \\ \hline \end{array}$$

$$\begin{array}{r} - 32x^5 \text{ Fifth power.} \\ \hline \end{array}$$

and $x^1 \times 5 = x^5$ the letter.

Therefore $-32x^5$ the fifth power required.

3. Required the 6th power of $\frac{a}{b^2}$ and the cube of

$$\frac{3x}{4y}$$

$$\text{Ans. } \left[\frac{a}{b^2} \right]^6 = \frac{a^1 \times 6}{b^2 \times 6} = \frac{a^6}{b^{12}} \text{ the 6}^{\text{th}} \text{ power of } \frac{a}{b^2}.$$

$$\text{And } \left[\frac{3x}{4y} \right]^3 = \frac{3x}{4y} \times \frac{3x}{4y} \times \frac{3x}{4y} = \frac{27x^3}{64y^3} \text{ the cube of } \frac{3x}{4y}.$$

4. Required the 3^d power of ab^2 and the 4th power of x^2y .

$$\text{Ans. } a^3b^6 \text{ and } x^8y^4.$$

5. Involve xy^2z^3 to the square and $-3x^2y^3$ to the cube.

$$\text{Ans. } x^2y^4z^6 \text{ and } -27x^6y^9.$$

6. Find the 7th power of $-\frac{3a^2x^3}{5y}$.

$$\text{Ans. } -\frac{2187a^{14}x^{21}}{78125y^7}.$$

7. Involve $a-x$ to the 5th power.

$a-x$ the root or 1st power.

$$a-x$$

$$\begin{array}{r} a^2 - a x \\ - a x + x^2 \\ \hline \end{array}$$

$a^2 - 2a x + x^2$ Square or 2^d power.

$$a-x$$

$$\begin{array}{r} a^3 - 2a^2 x + a x^2 \\ - a^2 x + 2a x^2 - x^3 \\ \hline \end{array}$$

$a^3 - 3a^2 x + 3a x^2 - x^3$ Cube or 3^d power.

$$a-x$$

$$\begin{array}{r} a^4 - 3a^3 x + 3a^2 x^2 - a x^3 \\ - a^3 x + 3a^2 x^2 - 3a x^3 + x^4 \\ \hline \end{array}$$

$a^4 - 4a^3 x + 6a^2 x^2 - 4a x^3 + x^4$ Biquadrate or 4th power.

$$a-x$$

$$\begin{array}{r} a^5 - 4a^4 x + 6a^3 x^2 - 4a^2 x^3 + a x^4 \\ - a^4 x + 4a^3 x^2 - 6a^2 x^3 + 4a x^4 - x^5 \\ \hline \end{array}$$

$a^5 - 5a^4 x + 10a^3 x^2 - 10a^2 x^3 + 5a x^4 - x^5$ 5th power.

8. Required the square of $x+y$, and the cube of $a-b$.

Ans. The square of $x+y$ is $x^2 + 2xy + y^2$, and the cube of $a-b$ is $a^3 - 3a^2 b + 3ab^2 - b^3$.

9. Involve $a+2x$ to the 4th power.

Ans. $x^4 + 8a^3 x + 24a^2 x^2 + 32a x^3 + 16x^4$.

10. What is the 3^d power of $x+1$ and of $x-1$?

Ans. $x^3 + 3x^2 + 3x + 1$ and $x^3 - 3x^2 + 3x - 1$.

NEWTON'S RULE FOR INVOLUTION. 101

11. Involve $a+b$ and $a-b$ each to the 4th power.

$$\text{Ans. } \begin{cases} a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \\ a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4 \end{cases}$$

12. Involve $-2a^3b^4$ to the square, $\frac{2ax^2}{3y^3}$ to the cube, $-\frac{ax}{2z^2}$ to the 4th power, and $x-y$ to the 5th power.

$$\text{Ans. } 4a^6b^8 \dots \frac{8a^3x^6}{27y^9} \dots \frac{a^4x^4}{16z^4}, \text{ and } x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5.$$

*SIR ISAAC NEWTON'S RULE for finding any power of a binomial or residual quantity more readily than by the common method of involution.

For the terms and indices.

1. Write down the first or leading quantity as many times as there are units in the index of the power to which it is to be involved.

2. Over the first of these place the index of the power, over the second the index decreased by 1,

* This rule expressed in general terms is as follows:

$$\overline{a+b}^n = a^n \pm na^{n-1}b \pm n \cdot \frac{n-1}{2} a^{n-2}b^2 \pm n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} a^{n-3}b^3 \pm n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} a^{n-4}b^4 \pm, \text{ \&c.}$$

The celebrated Sir Isaac Newton discovered this rule by induction: since his time several mathematicians have given demonstrations of it, which depend chiefly on figurate numbers, or the doctrine of fluxions: Landen, Ronayne, Masères, and some others have demonstrated the rule by pure algebra, but their methods are mostly too intricate to be given in this place.

102 NEWTON'S RULE FOR INVOLUTION.

over the third the index decreased by 2, and so on, making the index of each term 1 less than that of the preceding.

Take now the other quantity; and beginning at the second term, annex it to that and the following terms of the leading quantity, and carry it one place beyond; let 1 be made the index of the first, 2 of the second, 3 of the third, and so on, increasing by 1 in every term to the last, whose index will be that of the given power.

This rule being observed, the indices will stand in the following order—

Indices of the leading quantity	5	4	3	2	1	.
Those of the following	1	2	3	4	5	

For the coefficients.

The coefficient of the first term is 1, and in any term if the coefficient be multiplied into the index of its leading quantity, and the product divided by the number denoting the place of that term, the quotient will be the coefficient of the next term.

For the signs.

When both terms of the root are +, all the terms of the power will be +; but when the second term is —, all the odd terms (viz. the 1st, 3^d, 5th, &c.) will be +, and the even terms (viz. the 2^d, 4th, 6th, &c.) minus.

Note. The number of terms in any power will be one more than the index of that power. The coefficients will successively increase up to the middle term, and then decrease by the same differences down to the last; so that the coefficients of the first and last terms will be alike, being each 1; those of the second and

NEWTON'S RULE FOR INVOLUTION. 103

last but one, will be alike, being each the index of the power; in like manner the coefficients of the third and last but two, the fourth and last but three, &c. to the middle term, will be respectively alike.

EXAMPLES.

1. Involve $a+x$ to the 4th power by Sir Isaac Newton's rule.

The leading quantity a with its indices will be . . . } a^4, a^3, a^2, a
 And the following quantity x will be } x, x^2, x^3, x^4
 These connected according to the rule, the terms without the coefficients will be } $a^4, a^3x, a^2x^2, ax^3, x^4$

Now to find the coefficients, that of the first term is 1 (understood) and its index 4, and being the first term, 1 denotes its place, therefore

$$\frac{1 \times 4}{1} = 4 \text{ is the coefficient of the second term.}$$

And multiplying this coefficient into the index 3 of the leading quantity a in the second term, and dividing by the number 2, denoting the place of that term, we have

$$\frac{4 \times 3}{2} = 6 \text{ the coefficient of the third term.}$$

And multiplying this coefficient into the index of a in the third term, viz. 2, and dividing the product by the number 3, denoting the place of the term, we have

$$\frac{6 \times 2}{3} = 4 \text{ the coefficient of the fourth term.}$$

104 NEWTON'S RULE FOR INVOLUTION.

In like manner 4 multiplied into 1, the index of a , in the fourth term, and the product divided by the number 4, denoting its place, we have

$$\frac{4 \times 1}{4} = 1 \text{ the coefficient of the last term (understood).}$$

And because both terms of the root are + all the terms of its power will be +.

Therefore the 4th power of $a+x$ is $a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4$.

2. Involve $a+b$ to the 5th power.

The leading quantity } a^5, a^4, a^3, a^2, a
 a will be
 And the following } b, b^2, b^3, b^4, b^5
 quantity
 These connected, the } $a^5, a^4b, a^3b^2, a^2b^3, ab^4, b^5$
 terms without the }
 coefficients will be }

1 (understood) is the coefficient of the first term.

$$\frac{1 \times 5}{1} = 5 \text{ the coefficient of the second term.}$$

$$\frac{5 \times 4}{2} = 10 \text{ the coefficient of the third term.}$$

$$\frac{10 \times 3}{3} = 10 \text{ the coefficient of the fourth term.}$$

$$\frac{10 \times 2}{4} = 5 \text{ the coefficient of the fifth term.}$$

$$\frac{5 \times 1}{5} = 1 \text{ the coefficient of the last term (understood).}$$

Therefore the 5th power of $a+b$ is

$$a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

NEWTON'S RULE FOR INVOLUTION. 105

3. Involve $x-z$ to the 6th power.

The terms without the coefficients will be

$$x^6, x^5z, x^4z^2, x^3z^3, x^2z^4, xz^5, z^6,$$

And the coefficients

$$1, \frac{1 \times 6}{1}, \frac{6 \times 5}{2}, \frac{15 \times 4}{3}, \frac{20 \times 3}{4}, \frac{15 \times 2}{5}, \frac{6 \times 1}{6},$$

$$\text{or, } 1, 6, 15, 20, 15, 6, 1.$$

And the signs will be alternately + and -.

Therefore the 6th power of $x-z$ is

$$x^6 - 6x^5z + 15x^4z^2 - 20x^3z^3 + 15x^2z^4 - 6xz^5 + z^6.$$

4. Involve $a+b$ to the cube or 3^d power.

$$\text{Ans. } a^3 + 3a^2b + 3ab^2 + b^3.$$

5. What is the 4th power of $a-y$?

$$\text{Ans. } a^4 - 4a^3y + 6a^2y^2 - 4ay^3 + y^4.$$

6. Required the 5th power of $a-z$.

$$\text{Ans. } a^5 - 5a^4z + 10a^3z^2 - 10a^2z^3 + 5az^4 - z^5.$$

7. Involve $m+n$ to the 6th power.

$$\text{Ans. } m^6 + 6m^5n + 15m^4n^2 + 20m^3n^3 + 15m^2n^4 + 6mn^5 + n^6.$$

8. Involve $c-z$ to the 7th power.

$$\text{Ans. } c^7 - 7c^6z + 21c^5z^2 - 35c^4z^3 + 35c^3z^4 - 21c^2z^5 + 7cz^6 - z^7.$$

EVOLUTION.

Evolution is the reverse of involution, and teacheth to find the roots of given quantities.

1. *To find the roots of simple quantities.*

RULE I*.

Extract the root of the coefficient for the numeral part, and multiply the indices of the letters into the index of the root; the products placed over the letters to which they severally belong, will give the literal part of the root, to which the numeral part being annexed, the root is obtained.

EXAMPLES.

1. Extract the square root of $4a^2$ and the cube root of $8x^3$.

$$\text{Ans. } \sqrt{4a^2} = 2a^{2 \times \frac{1}{2}} = 2a^1 = 2a.$$

$$\text{and } \sqrt[3]{8x^3} = 2x^{3 \times \frac{1}{3}} = 2x^1 = 2x.$$

Explanation.

In the first place the square root of 4 is 2 for the coefficient of the root; secondly, the index 2 being multiplied into the index of the square root, viz. $\frac{1}{2}$ gives 1 for the index of a , the root therefore is $2a^1$ or $2a$, the index 1 being always understood.

In like manner the cube root of 8 is 2, and the index 3 being multiplied into $\frac{1}{3}$, which is the index of the cube root, gives 1 for the index of x , the cube root of $8x^3$ is therefore $2x$.

* This rule being the reverse of involution, the reasons on which its operations are founded will be sufficiently obvious.

2. Extract the square root of $9a^4$ and the cube root of $64z^9$.

$$\text{Ans. } \sqrt{9a^4} = 3a^{4 \times \frac{1}{2}} = 3a^2 = 3a^2.$$

$$\text{and } \sqrt[3]{64z^9} = 4z^{9 \times \frac{1}{3}} = 4z^3 = 4z^3.$$

3. Extract the 4th root of $16a^6x^5$.

$$\text{Ans. } \sqrt[4]{16a^6x^5} = 2a^{\frac{6}{4}}x^{\frac{5}{4}} = 2a^{\frac{3}{2}}x^{\frac{5}{4}}.$$

4. Required the square root of $\frac{x^2y^3}{a^4z^6}$.

$$\text{Ans. } \sqrt{\frac{x^2y^3}{a^4z^6}} = \frac{xy^{\frac{3}{2}}}{a^2z^3}.$$

5. Required the square root of $9x^2$ and the cube root of $8a^3$.

$$\text{Ans. } 3x \text{ and } 2a.$$

6. Find the cube root of $-64x^3y^6$.

$$\text{Ans. } -4xy^2.$$

7. Extract the square root of $81x^4y^8$ and the cube root of $27a^3b^6$.

$$\text{Ans. } 9x^2y^4 \text{ and } 3ab^2.$$

8. To find the 4th root of $81y^4z$ and the cube root of a^3x .

$$\text{Ans. } 3yz^{\frac{1}{4}} \text{ and } ax^{\frac{1}{3}}.$$

9. Extract the cube root of $\frac{8a^6}{27z^9}$ and the square root of $-7a^2x^4$.

$$\text{Ans. } \frac{2a^2}{3z^3}, \text{ the square root of } -7a^2x^4 \text{ is impossible.}$$

RULE 2.

If the given quantity be not a complete power, divide it by the greatest power that will divide it; place the quotient under the radical sign, and then annex it to the root of that power.

EXAMPLES.

1. Extract the square root of
- $2a^2x$
- .

$$\text{Ans. } \sqrt{2a^2} = \sqrt{a^2 \times 2} = a\sqrt{2}.$$

Explanation.

Here a^2 is the greatest square quantity in $2a^2$, and dividing this latter by it the quotient is 2, which put under the radical sign, and annexed to the root a is $a\sqrt{2}$.

2. Required the square root of
- $12a^3$
- and the cube root of
- $3x^3$
- .

$$\text{Ans. } \sqrt{12a^3} = \sqrt{4a^2 \times 3a} = 2a\sqrt{3a}.$$

$$\text{and } \sqrt[3]{3x^3} = \sqrt[3]{x^3 \times 3} = x\sqrt[3]{3}.$$

3. Extract the square root of
- $3x^2y^4$
- and the cube root of
- $3ax$
- .

$$\text{Ans. } xy^2\sqrt{3} \text{ and } \sqrt[3]{3ax}.$$

4. What is the square root of
- $\frac{9x^2y^2}{4a^3}$
- ?

$$\text{Ans. } \frac{3xy}{2a\sqrt{a}}.$$

5. Extract the cube root of
- $\frac{3x^3y^6}{4a^6z^3}$
- .

$$\text{Ans. } \frac{xy^2}{a^2z}\sqrt[3]{\frac{3}{4}}.$$

6. Required the 4
- th
- root of
- $32a^4x$
- and the 5
- th
- root of
- ay^2
- .

$$\text{Ans. } 2a\sqrt[4]{2x} \text{ and } \sqrt[5]{ay^2}.$$

2. To extract the square root of a compound quantity.

RULE*.

Range the quantities according to the dimensions of some letter as in division.

Find the root of the first term and set it in the quotient, and its square under the first term, and subtract.

Bring down the two next terms to the remainder for a dividend, and set double the root on its left for a divisor.

Divide the dividend by the divisor, and annex the result to both the quotient and divisor.

Multiply the divisor so increased, by the term last put in the quotient; subtract the product from the dividend, and bring down the two next terms to the remainder for a new dividend.

Bring down the divisor, and having doubled its last term, place the result to the left of the new dividend for a divisor, divide, annex the result to both quotient and divisor—multiply—subtract—bring down two terms, &c. as before—proceed in this manner till the work is finished.

* The method by which roots are extracted consists entirely in employing a process the reverse of involution, and which was evidently pointed out by it: thus, if $a+x$ be a root of which the part a is known and x unknown; then the square of this root being $a^2+2ax+x^2$, which is the composition, this indicated, the method of resolution so as to find out the unknown part x ; for having subtracted the nearest square a^2 from the given quantity, there remains $2ax+x^2$ or $(2a+x) \times x$; therefore dividing this remainder by $2a$, double of the first member of the root, the quotient will be nearly x the other member; then to $2a$, adding this quotient x , and multiplying the sum $2a+x$ into x , the product will make up the remaining part $2ax+x^2$ of the given power; wherefore the root $a+x$ is found, and the method by which it is obtained agrees exactly with the above rule; much in the same manner are the rules for extracting the cubic and other roots derived.

Note. The operations in this and the following problem may be proved, either by addition or by involving the root to the given power.

EXAMPLES.

1. Extract the square root of $a^4 + 4a^3 + 6a^2 + 4a + 1$.

$$a^4 + 4a^3 + 6a^2 + 4a + 1 \text{ (} a^2 + 2a + 1 \text{ root.)}$$

$$a^4 \dots \dots \dots = \text{the sq. of } a^2.$$

$$2a^3 + 2a \dots 4a^3 + 6a^2 \dots \dots \dots 1^{\text{st}} \text{ dividend.}$$

$$4a^3 + 4a^2 \dots \dots \dots = (2a^2 + 2a) \times 2a.$$

$$2a^2 + 4a + 1 \dots \dots 2a^2 + 4a + 1 \text{ } 2^{\text{d}} \text{ dividend.}$$

$$2a^2 + 4a + 1 = (2a^2 + 4a + 1) \times 1$$

$$\dots \dots \dots$$

$$a^4 + 4a^3 + 6a^2 + 4a + 1 \text{ proof by addition.}$$

Explanation.

First, The greatest square in a^4 is a^2 ; I put its root a^2 in the quotient, and subtract the square, bringing down the two next terms $4a^3 + 6a^2$; I then double the root a^2 and it becomes $2a^2$, which is placed for a divisor; and $4a^3$ divided by it, the quotient of which is $2a$, this is placed both in the quotient and divisor; I now multiply the divisor $2a^2 + 2a$ into $2a$, the result is $4a^3 + 4a^2$, this is placed under the dividend, and subtracted from it, and the two next terms are brought down to the remainder for a new dividend; I next take the divisor $2a^2 + 2a$, and

doubling its last line it becomes $2x^2 + 4a$, and dividing the new dividend $2a^2 + 4a + 1$ by it, the result is 1, which placed both in the quotient and divisor, and multiplied into the latter, produces the last line $2a^2 + 4a + 1$. The addition proof done the same way as in division.

2. Extract the square root of $x^4 - 6x^3 + 13x^2 - 12x + 4$.

$$\begin{array}{r}
 x^4 - 6x^3 + 13x^2 - 12x + 4 \quad (x^2 - 3x + 2 \text{ the rt.} \\
 x^4 \\
 \hline
 2x^3 - 3x^2 \quad -6x^3 + 13x^2 \\
 \quad \quad -6x^3 + 9x^2 \\
 \hline
 2x^2 - 6x + 2 \quad 4x^2 - 12x + 4 \\
 \quad \quad 4x^2 - 12x + 4 \\
 \hline
 \quad \quad * \quad * \quad * \\
 \hline
 \end{array}$$

Proof by involution.

$$\begin{array}{r}
 x^2 - 3x + 2 \\
 x^2 - 3x + 2 \\
 \hline
 x^4 - 3x^3 + 2x^2 \\
 \quad - 3x^3 + 9x^2 - 6x \\
 \quad \quad 2x^2 - 6x + 4 \\
 \hline
 x^4 - 6x^3 + 13x^2 - 12x + 4 \\
 \hline
 \end{array}$$

L 2

Subtract its power from that term, and bring down the second term for a dividend.

Involve the root last found to the next lower power, and multiply the result into the index of the given power for a divisor.

Divide the dividend by the divisor, and the quotient will be the next term of the root.

Involve the whole root to the given power, subtract it from the top line, and divide the remainder as before, for another term of the root.

Involve the whole root, subtract, divide, &c. as before till nothing remains, and the work is finished.

EXAMPLES.

1. Extract the cube root of $x^6 + 6x^3 + 15x^4 + 20x^3 + 15x^2 + 6x + 1$.

$$\begin{array}{r} x^6 + 6x^3 + 15x^4 + 20x^3 + 15x^2 + 6x + 1 \quad (x^2 + 2x + 1 \text{ the root.}) \\ x^6 \dots \dots \dots \text{the cube of } x^2. \end{array}$$

$3x^4) \dots 6x^3 \dots \dots \dots$ the 2^d term divided by $3x^2$.

$$\begin{array}{r} x^6 + 6x^3 + 12x^4 + 8x^3 \dots \dots \text{the cube of } x^2 + 2x. \end{array}$$

$3x^4) \dots \dots \dots 3x^4 \dots \dots \dots$ the rem. divided by $3x^2$.

$$\begin{array}{r} x^6 + 6x^3 + 12x^4 + 20x^3 + 15x^2 + 6x + 1 \quad \text{the cube of } x^2 + 2x + 1. \end{array}$$

Extract the roots of some of the most simple terms, and connect them together by the sign + or - as may be judged most suitable for the purpose.

Involve the compound root thus found, to the proper power, and if it be the same with the given quantity, the compound root is the root required.

But

Explanation.

Here x^3 being the cube root of the first term, is placed in the quotient, and its cube, viz. x^6 under the first term, then $6x^5$, the second term, is brought down for a dividend, and the root x^2 is involved to that power, which is next lower than the cube that is to the square, and multiplied into 3, the index of the cube for a divisor, see the work $x^2 \times x^2 \times 3 = 3x^4$ the divisor. . . next $6x^5$ divided by $3x^4$ gives $2x$ for the second term of the root, the whole of which, viz. $x^2 + 2x$, is now involved to the cube, as in the third line of the work, and this subtracted from the top line, the first term of the remainder is $3x^4$, which being divided by the common divisor $3x^4$, gives 1 for the third term of the root. The whole root $x^2 + 2x + 1$ is now involved to the cube, which is the last line of the work, and being exactly like the top line, shews that $x^2 + 2x + 1$ is the required root.

But if it differs in some of the signs only, change them from + to -, or from - to +, till its power agrees with the given one throughout.

Thus in the 8th example, the root $2x - y$ is the difference of the roots of the first and last terms, and the same may be observed of the roots in examples 3, 6 and 10.

In example 4th the root $x + y + z$ is the sum of the roots of the first, fourth and sixth terms, and the root of example 9 is the sum of the roots of the first and last terms.

Should the above problem be found too difficult for the learner, he may omit it for the present, as also any of the following promiscuous examples, the methods of extracting roots by approximation, which are shewn further on, being far more easy and practicable, though this problem has the advantage in point of exactness.

2. Extract the square root of $x^4 - 2x^3y + 3x^2y^2 - 2xy^3 + y^4$.

$$\begin{array}{r} x^4 - 2x^3y + 3x^2y^2 - 2xy^3 + y^4 (x^2 - xy + y^2, \\ x^4 \dots \dots \dots \text{the square } x^2. \end{array}$$

$2x^3$) .. $-2x^3y$.. the second term divided by $2x^2$.

$$\begin{array}{r} x^4 - 2x^3y + x^2y^2 \text{ the square of } x^2 - xy. \\ \hline \end{array}$$

$2x^2$) .. $2x^2y^2$.. the remainder divided by $2x^2$.

$$\begin{array}{r} x^4 - 2x^3y + 3x^2y^2 - 2xy^3 + y^4 \text{ the sq. of } x^2 - xy + y^2 \\ \hline \end{array}$$

The divisor is thus produced . . x^2 is already in the next lower power than its square, and multiplying it into 2, the index of the square, the result is $2x^2$ the divisor.

3. Extract the 4th root of $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$

$$\begin{array}{r} x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4 (x - y \\ x^4 \end{array}$$

$4x^3$) $-4x^3y$

$$\begin{array}{r} x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4 \\ \hline \end{array}$$

4. Extract the square root of $x^2 + 2xy + 2xz + y^2 + 2yz + z^2$. *Ans.* $x + y + z$.

5. Extract the cube root of $a^3 - 3a^2x + 3ax^2 - x^3$. *Ans.* $a - x$.

6. Required the cube root of $y^6 + 6y^5 - 4y^3 + 96y - 64$. *Ans.* $y^2 + 2y - 4$.

7. To find the cube root of $x^6 + 9x^5 + 33x^4 + 63x^3 + 66x^2 + 36x + 8$. *Ans.* $x^2 + 3x + 2$.

8. Required the biquadrate root of $16x - 32x^3y + 24x^2y^2 - 8xy^3 + 4$. *Ans.* $2x - y$.
 9. Extract the biquadrate root of $16y^4 + 96y^3z + 216y^2z^2 + 216yz^3 + 81z^4$. *Ans.* $2y + 3z$.
 10. Extract the 5th root of $32x^5 - 80x^4 + 80x^3 - 40x^2 + 10x - 1$. *Ans.* $2x - 1$.

PROMISCUOUS EXAMPLES FOR PRACTICE.

1. Find the square and square root of $4a^2x^4$.
Ans. The square is $16a^4x^8$ and the square root $2ax^2$.
 2. What is the cube and cube root of $8x^3y^6$?
Ans. The cube $512x^9y^{18}$, cube root $2xy^2$.
 3. Multiply the square root and cube root of $64a^6$ together.
Ans. $32a^5$.
 4. Required the difference of $\overline{x+y}^2$ and $\overline{x-y}^2$.
Ans. $4xy$.
 5. Required the difference of $\overline{x+z}^3$ and $\overline{x-z}^3$.
Ans. $6x^2z + 2z^3$.
 6. Find the 4th power and 4th root of $2a^4x$.
Ans. The 4th power $16a^{16}x^4$. . the 4th root $a\sqrt[4]{2x}$.
 7. What is the quotient of the square of $4a^4x^2$ divided by half its square root? *Ans.* $16a^6x^3$.
 8. Multiply the square root of $4a^2x^4$ by three times its square.
Ans. $96a^5x^{10}$.
 9. What is the 4th power of $2a^2$ and the 5th root of $32z^5$? *Ans.* $16a^8$ and $2z$.
 10. Multiply the square of x^2y^2 by its square root.
Ans. x^5y^5 .
 11. Divide the 5th power of x^4 by 10 times its square root.
Ans. $\frac{1}{10}x^{18}$.
 12. What is the difference of $x^2 + 2x + 1$ and its square root?
Ans. $x^2 + x$.

SIMPLE EQUATIONS.

An equation is an expression wherein two quantities are declared equal by means of the sign $=$ of equality placed between them.

Thus $5+3=8$ is an equation, denoting that 5 with 3 added to it equals 8. Also, $4-1=3$, and $3+2-1=4$, and $8-2=5+1$, are equations, each denoting that the quantity on one side the $=$ is equal to that on the other side.

The resolution of problems is performed by means of equations, each equation contains quantities, some of which are known, that is, their values are given and understood, others are unknown, and their values are required to be found. The reduction of an equation is a method of managing its terms so as to get the unknown quantity by itself on one side the $=$ and known ones only on the other; when this is done, the value of the unknown quantity is found, for it is equal to those which are known.

A simple equation is that which contains the unknown quantity in its first power only.

Reduction of simple equations is performed by the following rules.

RULE 1*.

Any quantity may be transposed, that is, removed from one side of the equation to the other, by changing its sign.

* If equal quantities be added to equals, the sums will be equal; and if equals be subtracted from equals, the remainders will be equal—these self evident truths being admitted, the reason of the above rule for transposition will readily appear.

Thus if $x - 4 = 2$

Then will $x = 2 + 4 = 6$, by transposing 4 and changing its sign from $-$ to $+$.

This being premised,

If the unknown quantity be affirmative, transpose all the known ones which are connected with it, to the other side, and collect them into one sum, which will be the answer required.

If the unknown quantity is negative, it must be transposed, so as to become affirmative, and the known quantities transposed on the other side, and collected as before.

EXAMPLES.

1. Given $x + 2 - 3 = 4$ to find the value of x .

Here $x + 2 - 3 = 4$ is given,
 which by transposing } $x + 2 = 4 + 3$
 -3 becomes . . . }
 and by transposing $+2$ } $x = 4 + 3 - 2$
 it is }
 which by collecting the }
 known quantities } $x = 5$ the answer.
 $+4 + 3$ and -2 }
 becomes }

For in example 1 transposing -3 is nothing more than adding $+3$ to each side of the equation; in like manner, transposing $+2$ is the very same as adding -2 to each side, thus:

The equation is $x + 2 - 3 = 4$

Add to each side $-2 + 3$ thus $-2 + 3 = -2 + 3$

The sum is $x = 4 - 2 + 3 = 5$ the same as
 wherefore the rule is evident. the example

2. Given $x-3+4-5=6$ to find the value of x .

Here are given $x-3+4-5=6$
 and transposing -5 it } $x-3+4=6+5$
 becomes }
 and transposing $+4$. . . $x-3=6+5-4$
 and by transposing -3 . . $x=6+5-4+3$
 Lastly, by collecting the }
 known quantities into } $x=10$ the answer.
 one sum, we have . . }

3. Given $2x-4=x+8$ to find x .

Here we have given $2x-4=x+8$
 which by transposing -4 } $2x=x+8+4$
 becomes }
 And by transposing $+x$ } $2x-x=8+4$
 we have }
 but $2x-x$ is equal to x , } $x=8+4$
 therefore }
 That is, by collecting the }
 known quantities into } $x=12$ the answer.
 one sum, }

4. Given $x+3-4=5$ to find the value of x .

We have here given $x+3-4=5$
 which by transposing -4 } $x+3=5+4$
 becomes }
 And transposing $+3$ it } $x=5+4-3$
 becomes }
 Lastly by summing up the }
 known quantities, we } $x=6$
 have }

5. Given $x-8=2$ to find the value of x .

Ans. $x=10$.

6. Given $x+4=25$ to find the value of x .

Ans. $x=21$.

7. Given $2x-12=x+2$ to find x .

Ans. $x=14$.

8. Given $3x+1=2x+5$ to find x .

Ans. $x=4$.

9. Given $x+8-2=3$ to find x .

Ans. $x=-3$.

RULE 2*.

If the unknown quantity be divided by any quantity, that divisor may be taken away, provided you multiply all the other terms of the equation into it.

Thus if $\frac{x}{2}=8$

Then will $x=8 \times 2=16$

And if $\frac{x}{3}=3+2$

Then will $x=9+6=15$

EXAMPLES.

1. Given $\frac{x}{3}+4=6$ to find the value of x .

Here we have given $\frac{x}{3}+4=6$.

First, by transposing $+4$, we have $\frac{x}{3}=6-4$.

And by taking away the divisor, }
and multiplying the other quan- } $x=18-12$.

ties $6-4$ into it, we have }
That is, by collecting the known } $x=6$ the answer.
quantities - - - - - }

* If equal quantities be multiplied into equals, the products will be equal; this principle being admitted, the above rule, which is a practical application of it, will be evident.

SIMPLE EQUATIONS.

121

2. Given $\frac{x}{2} - 8 + 3 = 1$ to find the value of x .

Here we have given $\frac{x}{2} - 8 + 3 = 1$

And by transposing $+3$ } $\frac{x}{2} - 8 = 1 - 3$
we have

In like manner by trans- } $\frac{x}{2} = 1 - 3 + 8$
posing -8 , we shall }
have

And by taking away the }
divisor 2, and multi- } $x = 2 - 6 + 16$
plying all the other }
terms into it, we have

And by collecting the } $x = 12$ the answer.
known quantities . }

*3. Given $\frac{x}{7} - 3 + 4 = 5$ to find x .

We have here given $\frac{x}{7} - 3 + 4 = 5$

And by transposing -3 } $\frac{x}{7} = 5 - 1 = 4$
 $+4$, which is equal }
to $+1$, we have . . }

And taking away the di- }
visor and multiplying } $x = 28$
it into the other term, }
we have

* We have hitherto transposed the terms one at a time; but as the learner is now supposed to be tolerably conversant with the method, it will be better to transpose all the terms at once, thus, if $x + 4 - 5 + 6 = 7$ then $x = 7 - 4 + 5 - 6 = 2$.

SIMPLE EQUATIONS.

4. Given $\frac{x}{8} - 8 + 7 - 6 + 5 = 4$ to find x .

Here, by transposing all
the known terms which
are on the first side of
the equation, and sum-
ming them up, we have $\left. \begin{array}{l} \frac{x}{8} = 4 + 8 - 7 + 6 - 5 \\ \phantom{\frac{x}{8}} = 6 \end{array} \right\}$

That is $\frac{x}{8} = 6$

And taking away the di-
visor 8 and multiply-
ing 6 into it $\left. \begin{array}{l} \phantom{\frac{x}{8}} \\ \phantom{\frac{x}{8}} \end{array} \right\} x = 48$

5. Given $\frac{x}{2} - 3 = 4$ to find the value of x .

Ans. $x = 14$.

6. Given $\frac{x}{3} + 4 = 5$ to find x .

Ans. $x = 3$.

7. Given $\frac{x}{4} + 3 - 2 = 4$ to find x .

Ans. $x = 12$.

RULE 3*.

When the unknown quantity is multiplied into any quantity, this multiplier may be taken away, provided you divide all the other terms of the equation by it.

Thus, if $3x + 6 = 9$

Then will $x + 2 = 3$ by taking away the multiplier 3, and dividing the other terms 6 and 9 by it.

* Like the foregoing rules this agrees with a self-evident principle, namely, "If equals be divided by equals, the quotients will be equal," of which the rule is a practical application.

SIMPLE EQUATIONS.

123

EXAMPLES.

1. Given $4x-8=16$ to find the value of x .

First, by transposing -8 we have $4x=16+8=24$

And dividing by 4 there arises $x=\frac{24}{4}=6$ the answer.

2. Given $\frac{2x}{3}+4=8$ to find the value of x .

Here, by transposing $+4$, } $\frac{2x}{3}=8-4=4$
we have }

And multiplying into 3 . . $2x=12$

And dividing by 2 . . . $x=6$ the answer.

3. Given $\frac{2x-3}{4}-1=2$ to find x .

First, transposing -1 } $\frac{2x-3}{4}=2+1=3$
there arises }

And multiplying into 4 } $2x-3=12$
the divisor of $2x-3$ }

Then transposing -3 . . $2x=12+3=15$

Lastly, dividing by 2 . . $x=7\frac{1}{2}$ the answer.

4. Given $2x-2=2$ to find the value of x .

Ans. $x=2$.

5. Given $\frac{4x}{5}-6=10$ to find x .

Ans. $x=20$.

6. Given $\frac{7x+8}{9}+10=11$ to find x .

Ans. $x=\frac{1}{7}$.

RULE 4*.

If the unknown quantity be under a radical sign, or have a fractional index. Transpose all the other terms, and take away the multiplier or divisor from the unknown quantity (if there are such) by the former rules, then involve each side to the power denoted by the radical sign or index.

Note. If the multiplier or divisor be without the radical sign, they must be taken away before the involution is made; but if under the radical sign, they must be taken away after the involution.

EXAMPLES.

1. Given $2\sqrt{x}+3=9$ to find the value of x .

First, transposing $+3$ we } $2\sqrt{x}=9-3=6$.
have

And dividing by 2 . . . $\sqrt{x}=\frac{6}{2}=3$

And involving both sides } $x=3^2=9$ the ans.
to the square to take
off the radical sign .

2. Given $\frac{3\sqrt[3]{x}}{4}+5=8$ to find x .

Here, transposing $+5$, } $\frac{3\sqrt[3]{x}}{4}=8-5=3$
we get

And multiplying into 4 . $3\sqrt[3]{x}=12$

* That the like powers of equal quantities are equal is true, though, perhaps, it will not be self-evident to a learner; in that case he had better convince himself of its truth by involving two equal quantities differently expressed to their 2d, 3d, &c. powers, from which experiment the reason of the above rule will be plain.

SIMPLE EQUATIONS.

125

And dividing by 3 . . . $\sqrt[3]{x} = \frac{12}{3} = 4$

And by involution to the }
cube $x = 4^3 = 64$ the ans.

3. Given $\sqrt{3x+4} = 7$ to find x .

First, by transposing $+4$, }
we obtain $\sqrt{3x} = 7 - 4 = 3$

And involving to the square $3x = 3^2 = 9$

Now dividing by 3 . . . $x = \frac{9}{3} = 3$ the ans.

4. Given $\sqrt{x-9} = 1$ to find the value of x .

Ans. $x = 100$.

5. Given $\frac{2\sqrt{x}}{3} + 4 = 6$ to find x .

Ans. $x = 9$.

6. Given $\sqrt{x+1} - 2 = 3$ to find x .

Ans. $x = 24$.

7. Given $\sqrt{4x-3} = 1$ to find x .

Ans. $x = 4$.

RULE 5*.

If that side of the equation which contains the unknown quantity be a complete power, extract the root of that power on each side of the equation.

Thus, if $4x^2 = 16$

Then will $\sqrt{4x^2} = \sqrt{16}$

That is $2x = 4$

And $x = 2$.

* That the like roots of equal quantities are equal to each other, appears by the foregoing note.

EXAMPLES.

1. Given
- $9x^2 - 3 = 22$
- to find the value of
- x
- .

Here, by transposing -3 ,
we have } $9x^2 = 22 + 3 = 25$

And extracting the square
root of each side the
equation } $3x = \sqrt{25} = 5$

And by division $x = \frac{5}{3} = 1\frac{2}{3}$ the anf.

2. Given
- $x^3 + 8 = 35$
- to find
- x
- .

First, by transposing } $x^3 = 35 - 8 = 27$
+8

And extracting the cube
root } $x = \sqrt[3]{27} = 3$ the anf.

3. Given
- $y^2 + 6y + 9 = 64$
- to find
- y
- .

Here the square root of $y^2 + 6y + 9$ is $y + 3$

Therefore we have . . . $y + 3 = \sqrt{64} = 8$

And by transposition . . $y = 8 - 3 = 5$ the anf.

4. Given
- $x^2 - 5 + 2 = 97$
- to find
- x
- .
- Ans.*
- $x = 10$
- .

5. Given
- $3x^3 + 2 = 26$
- to find
- x
- .
- Ans.*
- $x = 2$
- .

6. Given
- $\frac{y^3}{4} - 5 = 11$
- to find
- y
- .
- Ans.*
- $y = 4$
- .

7. Given
- $3x^2 - 100 = 320687$
- to find
- x
- .
- Ans.*
- $x = 327$
- .

RULE 6*.

Any analogy or proportion may be converted into an equation, by making the product of the two mean terms equal to that of the two extremes.

* Thus $2 : 4 :: 6 : 12$ } are propor- } $2 \times 12 = 4 \times 6 = 24$
Also, $a : 5a :: 2b : 10b$ } tionals and } $a \times 10b = 5a \times 2b = 10ab$
wherefore the rule is evident.

SIMPLE EQUATIONS.

127

Thus, if $x : 5 :: 6 : 3$ Then will $3x = 5 \times 6 = 30$ And $\therefore x = \frac{30}{3} = 10.$

EXAMPLES.

1. Given
- $3y : 4 :: 5 : 3$
- to find the value of
- y
- .

First, by multiplying extremes and means, we } $3 \times 3y = 4 \times 5$
 have

That is $9y = 20$ And by division $y = \frac{20}{9} = 2\frac{2}{9}$ the ans.

2. Suppose
- $\frac{2x}{3} : 4 :: 6 : 7$
- were given to find the value of
- x
- .

Here, by multiplying extremes and means, we } $\frac{2x}{3} \times 7 = 4 \times 6$
 have

That is $\frac{14x}{3} = 24$ And multiplying into 3 $14x = 72$ Lastly, dividing by 14 $x = \frac{72}{14} = 5\frac{2}{14} = 5\frac{1}{7}$ the ans.

3. Given
- $4z : 5 :: 14 : 7$
- to find
- z
- .

Here, multiplying extremes and means . . } $28z = 70$

And dividing by 28 . . . $z = \frac{70}{28} = 2\frac{1}{2}$ the ans.

4. Given $3 : 4 :: x : 5$ to find x . *Ans.* $x = 3\frac{3}{4}$.

5. Given $\frac{3x}{5} : 4 :: 2 : 1$ to find x . *Ans.* $x = 13\frac{1}{3}$.

6. Given $2\sqrt{x} : 14 :: 4 : 7$ to find x .
Ans. $x = 16$.

7. Given $\frac{x}{2} : \frac{2}{3} :: 3 : 2$ to find x . *Ans.* $x = 1$.

RULE 7*.

If the same quantity be on both sides of the equation, with the same sign, it may be taken away from each; and if every term of an equation be multiplied or divided by the same quantity, it may be struck out of them all.

Thus, if $2x + a = b + a$, the a being on both sides may be taken away, and we have $2x = b$ and $x = \frac{b}{2}$.

Also, if $ax = ab - ac$

Then will $x = b - c$

EXAMPLES.

1. Given $3x + 5 = 4 + 5$ to find x .

Here $+5$ is on both sides, and may be left out.

Therefore $3x = 4$

And $\therefore x = \frac{4}{3} = 1\frac{1}{3}$ the answer.

2. Given $4ax - 4ab = 4ac$ to find x .

Here, leaving out the common multiplier $4a$,

We have $x - b = c$

And transposing $-b$ $\therefore x = c + b$ the answer.

* The reason of this rule will appear from the notes on rules 1 and 2.

SIMPLE EQUATIONS.

129

FROMISCUOUS EXAMPLES.

1. Given $\frac{x}{2} + \frac{x}{3} + 4 = 5$ to find the value of x .

First, multiplying by 3, $\left\{ \begin{array}{l} \frac{3x}{2} + x + 12 = 15 \\ \text{we have} \dots \dots \dots \end{array} \right.$

And multiplying by 2, $3x + 2x + 24 = 30$

Transposing $+24$. $3x + 2x = 30 - 24 = 6$

That is . $5x = 6$

And . $x = \frac{6}{5} = 1\frac{1}{5}$.

2. Given $4y - 14 = 3y + 12$ to find y .

First, by transposing $4y - 3y = 12 + 14$

That is $y = 26$ the answer.

3. Let there be given $21 - 7x = 40 - 11x$ to find x .

First, by transposition $11x - 7x = 40 - 21$

That is $4x = 19$

And $x = 4\frac{3}{4}$ the answer.

4. Let there be given $3x^2 - 10x = x^2 + 2x$ to find x .

First, $3x - 10 = x + 2$

That is $2x = 12$

And $x = 6$ the answer.

5. What is the value of x when $\frac{2x}{3} + \frac{x}{4} = \frac{x}{2} + 4$.

First, $2x + \frac{3x}{4} = \frac{3x}{2} + 12$

Also, $8x + 3x = \frac{12x}{2} + 48$

SIMPLE EQUATIONS.

And $16x + 6x = 12x + 96$

That is $22x - 12x = 96$

Or, $10x = 96$

And $x = 9\frac{3}{5}$ the answer.

6. Given $\frac{x+1}{2} + \frac{x-1}{3} - 4 = \frac{x}{4} + 1$ to find x .

First, $x+1 + \frac{2x-2}{3} - 8 = \frac{x}{2} + 2$

And $3x+3+2x-2-24 = \frac{3x}{2} + 6$

That is $5x - \frac{3x}{2} = 29$

Also, $10x - 3x = 58$

And $x = 8\frac{2}{3}$.

7. Given $\sqrt{\frac{2x}{3}} + 5 = 7$ to find x .

First, $\sqrt{\frac{2x}{3}} = 7 - 5 = 2$

Whence $\frac{2x}{3} = 2^2 = 4$

And $2x = 12$

Wherefore $x = 6$ the answer.

8. Let $3x^2 + 4 = 196$ be given to find x .

First, $3x^2 = 196 - 4 = 192$

Also, $x^2 = \frac{192}{3} = 64$

And $x = \sqrt{64} = 8$ the answer.

9. Given $\frac{x}{2} + \frac{x}{3} + 4 = 14$ to find x . *Ans.* $x = 12$.

SIMPLE EQUATIONS.

131

10. Given $\frac{x}{2} - \frac{x}{3} + \frac{x}{4} = 10$ find x . *Ans.* $x=24$.

11. Given $40 - 6x = 136 - 14x$ to find x .

Ans. $x=12$.

12. Given $y + 12 = 3y - 4$ to find y . *Ans.* $y=8$.

13. Let $x + \frac{x}{2} + \frac{x}{3} = 22$ be given to find x .

Ans. $x=12$.

14. If $\frac{1}{2}y + \frac{1}{3}y - \frac{1}{4}y = \frac{1}{2}$ what is y . *Ans.* $y=\frac{6}{7}$.

15. Given $\frac{x^2}{4a} - b = 2c$ to find x .

Ans. $x = \sqrt{8ac + 4ab}$.

16. Suppose $2x^2 - 3x = x^2 + x$, what is x ?

Ans. $x=4$.

17. Given $\frac{4\sqrt{x}}{9} + 40 = 44$ to find x . *Ans.* $x=81$.

A collection of easy problems, whose solutions depend on the foregoing rules.

1. What number is that to which 5 being added the sum will be 40?

Let the number be denoted }
by } x

To which 5 being added, }
it becomes } $x+5$

But by the problem this sum }
is equal to 40, therefore } $x+5=40$

And by transposition $x=40-5=35$ the
number required.

2. What number is that from which 8 being subtracted the remainder is 45?

Let the number be denoted }
by } x

From which subtracting 8, $\left\{ \begin{array}{l} x-8 \end{array} \right.$
it becomes

The result will be 45, there- $\left\{ \begin{array}{l} x-8=45 \end{array} \right.$
fore

And by transposition $x=45+8=53$ the
number required.

3. What number is that, which being multiplied
into 3, with 4 added to the product, the sum will be 16?

Let the number be denoted $\left\{ \begin{array}{l} x \end{array} \right.$
by

Which multiplied into 3 $\left\{ \begin{array}{l} 3x \end{array} \right.$
is

To which adding 4, the $\left\{ \begin{array}{l} 3x+4 \end{array} \right.$
sum is

And by the problem this $\left\{ \begin{array}{l} 3x+4=16 \end{array} \right.$
sum is equal to 16, there-
fore

And by transposition . . . $3x=16-4=12$

And by division $x=\frac{12}{3}=4$ the num-
ber required.

4. What number is that which being multiplied
into 4, and the product divided by 5, and if 3 be sub-
tracted from the quotient, the remainder is 20?

Let the number be denoted $\left\{ \begin{array}{l} x \end{array} \right.$
by

Which multiplied into 4 is $4x$

This product divided by 5 $\left\{ \begin{array}{l} \frac{4x}{5} \end{array} \right.$
becomes

From which subtracting 3, $\left\{ \begin{array}{l} \frac{4x}{5}-3 \end{array} \right.$
the remainder is

And this is by the problem } $\frac{4x}{5} - 3 = 20$
 equal to 20, wherefore

And by transposition - - - $\frac{4x}{5} = 20 + 3 = 23$

And by multiplication - - $4x = 115$

And by division - - - - $x = \frac{115}{4} = 28\frac{3}{4}$ the
 number required.

5. What number is that to which 4 being added,
 and the sum divided by 6, the quotient is 12?

Let the number be repre- }
 sented by - - - - - } x

To which adding 4, it be- }
 comes - - - - - } $x + 4$

This sum divided by 6 is - $\frac{x+4}{6}$

Which, by the problem, is } $\frac{x+4}{6} = 12$
 equal to 12, that is - - }

And by multiplication - - - $x + 4 = 72$

And by transposition - - - $x = 72 - 4 = 68$ the
 number required.

6. What number is that from which 9 being sub-
 tracted, and the remainder multiplied into 11, the
 product is 121?

Let the number be repre- }
 sented by - - - - - } x

From which 9 being sub- }
 tracted, it becomes - - } $x - 9$

This multiplied into 11 is - $11 \times \overline{x-9}$

Which, by the problem, is } $11 \times \overline{x-9} = 121$
 equal to 121, that is - }

N.

EASY PROBLEMS.

This divided by 11 is - - - $x - 9 = \frac{121}{11} = 11$

And by transposition - - - $x = 11 + 9 = 20$ the number required.

7. A man being asked his age replied thus, if 3 times the square root of my age be subtracted from 30, the remainder is 12, what was his age?

Let his age be denoted by - x

Then will 3 times its square root be - - - - - $3\sqrt{x}$

This subtracted from 30 is - $30 - 3\sqrt{x}$

Which, by the problem, is equal to 12, wherefore we have - - - - - $30 - 3\sqrt{x} = 12$

And by transposition - - - $30 - 12 = 3\sqrt{x}$

That is - - - - - $3\sqrt{x} = 18$

And by division - - - - - $\sqrt{x} = \frac{18}{3} = 6$

Whence by involution - - $x = 6^2 = 36$ the ans.

8. What number is that to which 12 being added, and 8 subtracted from this sum, the remainder is 30?

This is done like prob. 1 and 2.

Ans. 26.

9. Required a number which if multiplied into 4, and 10 added to the product, the sum is 46.

Done like prob. 3.

Ans. 9.

10. Find a number which being divided by 6, with 2 subtracted from the quotient, the remainder is 2.

Done like prob. 4 nearly.

Ans. 24.

11. A man being asked how many shillings he had, answered, add 15 to their number, then subtract 1 from this sum, and the remainder will be 64, how many had he?

Ans. 50.

Done like prob. 1 and 2.

EASY PROBLEMS.

135

12. Take 4 years from my age, multiply the remainder into 5, and the product will be 165, how old am I?

Ans. 37 years.

Done like prob. 6.

13. There is a field to which 8 acres being added, and the sum divided by 9, the quotient is 3, how many acres does the field contain?

Ans. 19 acres.

Done like prob. 5.

14. In an orchard $\frac{1}{4}$ of the trees bear apples, $\frac{1}{5}$ pears, $\frac{2}{11}$ plumbs, and 81 cherries, how many trees are there, and how many of each sort?

Let the number of trees be denoted by x

Then will $\frac{1}{4}$, viz. those which bear } $\frac{x}{4}$
apples be - - - - -

And $\frac{1}{5}$ part, or the number of pear } $\frac{x}{5}$
trees - - - - -

And $\frac{2}{11}$, or the number of plumbs - $\frac{2x}{11}$

Also the cherries - - - - - 81

These added together will
give the whole number } $\frac{x}{4} + \frac{x}{5} + \frac{2x}{11} + 81 = x$
of trees in the orchard,
viz. - - - - -

And multiplying by 4 . . $x + \frac{4x}{5} + \frac{8x}{11} + 324 = 4x$

Ditto by 5 - - - - $5x + 4x + \frac{40x}{11} + 1620 = 20x$

Ditto by 11 - - $55x + 44x + 40x + 17820 = 220x$

Which, by addition and transpo- } $81x = 17820$
sing, becomes - - - - -

And by division - - - - - $x = \frac{17820}{81} = 220$

the number of trees required.

Also, $\frac{x}{4} = \frac{220}{4} = 55$ apple trees

And $\frac{x}{5} = \frac{220}{5} = 44$ pear trees

And $\frac{2x}{11} = \frac{440}{11} = 40$ plumbs

And - - - - - 81 cherries

220 proof.

15. In a certain school $\frac{1}{2}$ of the boys learn mathematics, $\frac{1}{3}$ study the classics, and 6 learn book-keeping, what number of scholars are there?

Ans. 120.

16. I have ordered my brewer to mix a quantity of beer for my use, $\frac{1}{2}$ of it is to be porter, $\frac{1}{3}$ ale, and 85 gallons of table beer, what quantity am I to have in the whole?

Ans. 120 gallons.

17. An estate consists of $\frac{1}{2}$ woodland, $\frac{2}{3}$ pasture, and 105 acres in pleasure grounds, gardens, and orchards, how many acres does it contain, and what the estate worth at 10 guineas and a half per acre?

Ans. 630 acres, . and its value £6945. 15s.

18. After paying away $\frac{1}{4}$ and $\frac{1}{3}$ of my money I found 22 guineas in my purse, how many had I at the first?

Ans. 40.

19. What number is that to which 10 being added $\frac{2}{3}$ of the sum will be 66?

Let the number be called - x

To which adding 10, it be- } $x + 10$
comes - - - - - }

$\frac{3}{5}$ of this sum is $\frac{3}{5} \times x + 10$ or $\frac{3x+30}{5}$

Which, by the prob. is equal } $\frac{3x+30}{5} = 66$
to 66, that is - - - - -

This reduced gives - - - - - $x = 100$ the number required.

20. A servant maid having received a legacy, expended £20. in cloaths, and small presents among her poor relations, and sent £108. which was $\frac{9}{10}$ of the remainder, to the bank, what was the legacy?

Ans. £140.

21. Bought sheep for £60. calves for £20. and pigs for £5. and laid out £3. which was $\frac{5}{8}$ of what I had left in necessary expences, how much money did I carry to market?

Ans. £88. 12s.

22. Of a piece of metal $\frac{1}{3}$ plus 24 ounces is brass, and $\frac{3}{4}$ minus 42 ounces copper, what is the weight of the piece?

Let the weight of the piece }
be - - - - - } x

Then will the weight of the } $\frac{x}{3} + 24$
brass be - - - - - }

And the weight of the copper $\frac{3x}{4} - 42$

These added together are } $\frac{x}{3} + \frac{3x}{4} + 24 - 42 = x$
equal to the whole piece, }
that is - - - - - }

Or, $\frac{x}{3} + \frac{3x}{4} - x = 18$

Which being reduced, we } $x = 216$ ounces, the
have - - - - - }
weight required.

The weight of the brass is $\frac{x}{3} + 24 = \frac{216}{3} + 24 = 96$ oz.

And that of the copper $\frac{3x}{4} - 42 = \frac{648}{4} - 42 = 120$ oz.

The whole piece = 216 proof

25. A farmer mixes a quantity of corn, $\frac{1}{2}$ minus 20 bushels is barley, and $\frac{1}{3}$ plus 36 bushels oats, how many bushels are there in the whole, and of each sort?

Ans 28 bush. of barley, 68 oats, and 96 bush. in the whole.

24. Of a detachment of foldiers $\frac{2}{3}$ are on actual duty, $\frac{1}{8}$ sick, $\frac{1}{3}$ of the remainder absent on leave, and there are 380 officers, required the number of men in the detachment?

Let the number of men be denoted by - x

Then $\frac{2}{3}$ on duty and $\frac{1}{8}$ sick, that is $\frac{2x}{3} + \frac{x}{8}$

These reduced to a com. denom. and added are $\frac{21x}{40}$

$\frac{1}{3}$ of the remainder is $\frac{1}{3}$ of $x - \frac{21x}{40}$ or $\frac{1}{3} \times \frac{40x - 21x}{40}$

Which, by reduction, is $\frac{19x}{200}$ the number of absentees.

Whence, by the problem, $\frac{21x}{40} + \frac{19x}{200} + 380 = x$

Which, by reduction, gives $x = 1000$ the number required.

23. A father dying left his son a fortune, $\frac{1}{3}$ of which he spent the first year, and $\frac{1}{4}$ of the remainder the year following, at the end of which he had 142 pounds left, how much was left him by his father?

Ans. £.1136.

26. A sheepfold was robbed three nights successively, the first night were taken $\frac{1}{4}$ and 2 sheep more, the second night $\frac{1}{3}$ wanting 2 sheep, and the third night 10 were stolen, after which it was found that $\frac{3}{4}$ of the whole flock remained, what number did it consist of?

Ans. 100.

27. A boy having stole apples in a farmer's orchard, was met by a servant who took half what he had; soon after another met him and took away half the remainder; lastly, the farmer sees him, from whom he narrowly escapes, leaving 10 behind, and getting clear off he has but one left, how many had he at first?

Ans. 44.

28. Divide 150 into 2 parts, in the ratio of 7 to 8.

Let the first be denoted by $-x$

Then will the other part be $150-x$

And by the problem $-x : 150-x :: 7 : 8$

And by multiplying extremes and means, we have $\left. \begin{array}{l} \text{ex-} \\ \text{tremes and means, we} \end{array} \right\} 8x = 1050 - 7x$

That is by transposition $-15x = 1050$

And $-x = \frac{1050}{15} = 70$ the first part.

Consequently $-150-x = 150-70=80$ the other part.

29. Divide £1234. between A and B, so that A's share may be to B's as 3 to 2.

Ans. A's share £740. 8s. B's £493. 12s.

30. Two persons buy a ship which cost £854. now the sum paid by A is to that paid by B, as 9 to 7, what sum did each contribute?

Ans. A paid £480. 7s. 6d. B £373. 12s. 6d.

31. A and B by trading together gain £100. of which A has £55. and B £45. now their joint stock in the beginning was £220. how much of this did each put in?

Ans. A put in £121. B £99.

32. A man bought the lease of a house, for which he agreed to pay as follows, $\frac{1}{4}$ of the money on taking possession; $\frac{1}{3}$ within two years after, and £250. at the end of the fourth year, what sum did he pay in all?

Ans. £600.

33. At a certain election 548 persons voted, and the successful candidate had a majority of 130, how many voted for each?

Ans. 339 and 209 respectively.

34. There is a fish whose tail weighs 9lb. his head weighs as much as his tail and half his body, and his body weighs as much as his head and his tail, what is the weight of the fish?

Let the weight of his body be x

Then will that of his head be $\frac{x}{2} + 9$

And by the latter part of } $x = \frac{x}{2} + 9 + 9$
the problem - - - - -

That is - - - - - $x = 36$ the weight
of his body.

And $\frac{x}{2} + 9$ (the wt of his } $= 27$
head) - - - - -

And his tail - - - - - 9

These added we have 72lb. for the wt.
of the fish.

EASY PROBLEMS.

141

35. Suppose the head of a fish to be 5 inches long, and his tail as long as his head and half his body, and his body as long as his head and tail together, required the length of the fish? *Ans.* 40 inches.

36. Divide the number 360 into 3 parts, that are to each other as 3, 4, and 5 respectively.

Let the first part be denoted by $3x$

Then will the second be - - $4x$

And the third - - - - - $5x$

And these added together

will, by the problem, } $12x = 360$

equal 360, that is - - }

Whence - - - $x = 30$

Consequently the first part - $3x = 90$

And the second, or - - - $4x = 120$

And the third, or - - - $5x = 150$

360 the proof.

37. A's age is double of B's, and B's is treble of C's, and the sum of all their ages is 70, what is the age of each? *Ans.* A's 42 years, B's 21, and C's 7.

38. Four persons rent a coal mine for £8500. per annum, now supposing their respective shares in it to be as the numbers 1, 2, 3, and 4, how much of the yearly rent must each pay?

Ans. A pays £850. B £1700. C £2550. and D £3400.

39. From London to Berwick is accounted 335 miles; now if a courier sets out from each of these places at the same time, and travels towards the other; the first at the rate of 3 miles an hour, and the other at 2, in what time will they meet? *Ans.* 67 hours.

40. Several persons betted a horse race, A lost £40. but B won a sum, which, if squared, and 100 subtracted from the square, dividing the remainder by 5, and subtracting 20 from the quotient, he could make up A's loss and be £100. in pocket, what sum did B win?

Ans. £30.

REDUCTION OF EQUATIONS *containing two unknown quantities.*

RULE I.

Find the value of that quantity which is least involved in each of the equations, by the preceding methods.

Put the two values thus found, equal to each other; this equation will contain but one unknown quantity, whose value may be found as before.

EXAMPLES.

1. Given $\begin{cases} 2x+3y=12 \\ 3x-2y=5 \end{cases}$ to find the values of x and y .

*From the first equation we have $x = \frac{12-3y}{2}$

* Here we find the value of x in each equation; the first is $\frac{12-3y}{2}$, and the second $\frac{5+2y}{3}$; these being each equal to x are equal to one another, and constitute a new equation, containing only one unknown quantity y , whose value is found easily enough; the value of x is obtained by substituting 6 in the place of $3y$, (y being found $=2$) by which means $\frac{12-3y}{2}$, the value of x , becomes $\frac{12-6}{2}=3$; this might have been done by using the second value

And from the second - - - $x = \frac{5+2y}{3}$

These quantities being equal to the same, (viz. to x) they are equal to one another, that is

$$\frac{5+2y}{3} = \frac{12-3y}{2}$$

That is, by multiplication, $10+4y=36-9y$
Whence, by transposition and division $y=2$

And $x = \frac{12-3y}{2} = \frac{12-6}{2} = \frac{6}{2} = 3.$

2. Given $\begin{cases} x+y=9 \\ 3x-2y=7 \end{cases}$ to find x and y .

From the first equation - - $x=9-y$

And from the second - - - $x = \frac{7+2y}{3}$

That is, by multiplication - $27-3y=7+2y$

And by transposition - - - $20=5y$

Whence - - $y=4$

And $x=9-y=9-4=5.$

In these examples the operation may be proved by substituting instead of the unknown quantities, their values in the given equations.

Thus $x+y=9$ becomes $5+4=9$
And $3x-2y=7$ becomes $15-8=7$ } the proof.

$\frac{5+2y}{3}$; which, by substituting for $2y$ it equals 4, becomes $\frac{5+4}{3}$
 $=\frac{9}{3}=3$, the very same as before.

3. Given $\begin{cases} \frac{x}{2} + \frac{y}{3} = 7 \\ \frac{x}{3} + \frac{y}{2} = 8 \end{cases}$ to find x and y .

From the first equation $x = 14 - \frac{2y}{3}$

And from the second $x = 24 - \frac{3y}{2}$

Therefore $14 - \frac{2y}{3} = 24 - \frac{3y}{2}$

And by multiplying $42 - 2y = 72 - \frac{9y}{2}$

Also $84 - 4y = 144 - 9y$

Whence, by transposition, $5y = 144 - 84 = 60$

And $y = \frac{60}{5} = 12$

And $x = 14 - \frac{2y}{3} = 14 - \frac{24}{3} = 14 - 8 = 6$.

4. Given $\begin{cases} x + y = 16 \\ x : y :: 3 : 1 \end{cases}$ to find x and y .

From the equation $x = 16 - y$

And from the analogy $x = 3y$

Therefore $16 - y = 3y$

Whence, by transposition } $4y = 16$

and division $y = 4$

And $y = 4$

And $x = 3y = 12$.

5. Given $4x + 6y = 46$ and $5x - 2y = 10$ to find x and y .

Ans. $x = 4$, $y = 5$.

6. Given $2x + 3y = 31$ and $4x - 3y = 17$ to find x and y .

Ans. $x = 8$, $y = 5$.

7. Given $3x+4y=38$ and $5x-6y=38$ to find x and y . Ans. $x=10$ $y=2$.

8. Given $4y+z=102$ and $y+4z=48$ to find y and z . Ans. $y=24$ $z=6$.

9. Given $\frac{x}{2} + \frac{y}{3} = 9$ and $x:y::4:3$ to find x and y . Ans. $x=12$ $y=9$.

RULE 2.

Find the value of one of the unknown quantities in that equation where it is least involved.

Substitute the value thus found, instead of that quantity, in the other equation, which will then contain but one unknown quantity to be found as before.

EXAMPLES.

1. Given $\begin{cases} x+y=8 \\ 2x+3y=19 \end{cases}$ to find x and y .

From the first equation $x=8-y$

And $2x=16-2y$

This substituted for $2x$ in the second equation there arises $16-2y+3y=19$

That is $y=19-16=3$

And $x=8-y=8-3=5$

2. Given $\begin{cases} 2x+3y=7 \\ 4x-5y=3 \end{cases}$ to find x and y .

From the first equation $x=\frac{7-3y}{2}$

And $4x=4 \times \frac{7-3y}{2} = \frac{28-12y}{2} = 14-6y$

Substituting this value for its equal $4x$ in the second equation, we have $14-6y-5y=3$

O

Whence, by transposition $11y=11$

Consequently $y=1$

$$\text{And } x = \frac{7-3y}{2} = \frac{7-3}{2} = \frac{4}{2} = 2.$$

3. Given $\begin{cases} x+2y=17 \\ 3x-y=2 \end{cases}$ to find x and y .

From the first equation . . . $x=17-2y$

And therefore $3x=3 \times 17-2y=51-6y$

This value substituted for $3x$ in the second equation, gives $\begin{cases} 51-6y-y=2 \end{cases}$

Which, by transposition, is $7y=49$

And $y=7$

Whence, also, $x=17-2y=17-14=3$.

4. Given $\begin{cases} x+y=8 \\ x:y::3:1 \end{cases}$ to find x and y .

From the equation $x=8-y$

This substituted for x in the analogy, we have . $\begin{cases} 8-y:y::3:1 \end{cases}$

And multiplying extremes and means $\begin{cases} 8-y=3y \end{cases}$

That is $4y=8$

And $y=2$

Whence . . $x=8-y=8-2=6$.

5. Given $2x-y=6$ and $4x+3y=22$ to find x and y .

Ans. $x=4$. $y=2$.

6. Given $3y-z=13$ and $3z-y=9$ to find y and z .

Ans. $y=6$. $z=5$.

7. Given $x-3y=80$ and $x:y::5:2$ to find x and y .

Ans. $x=200$. $y=40$.

8. Given $x+\frac{y}{2}=12$ and $y+\frac{x}{2}=9$ to find x and y .

Ans. $x=10$. $y=4$.

RULE 3.

Multiply or divide the given equations by such a quantity as will make the coefficient of one of the unknown quantities the same in both*.

Then by adding or subtracting the equations, as the case may require, there will arise a new equation with only one unknown quantity.

EXAMPLES.

1. Given $\begin{cases} 3x+2y=23 \\ 4x-7y=21 \end{cases}$ to find x and y .

The first equation multiplied into 4 is $12x + 8y = 92$
 And the second multiplied into 3 $12x - 21y = 63$

Subtract in order to exterminate x , and we have. $29y = 29$

Therefore $y = 1$

In like manner to exterminate y , we multiply the first equation into 7, and the second into 2, and thence arises

And $\begin{cases} 21x + 14y = 161 \\ 8x - 14y = 42 \end{cases}$

Now to exterminate y we must add these equations together, and $29x = 203$

And $x = \frac{203}{29} = 7$

* If the first equation be multiplied into the coefficient of the quantity you would exterminate in the second, and the second into the coefficient of that quantity in the first; the coefficients of the quantity to be exterminated will be the same in both the resulting equations.

2. Given $\begin{cases} 3x+5y=40 \\ x+2y=14 \end{cases}$ to find x and y .

Multiply the second eq. into 3, and it
will give $3x+6y=42$

Now subtract the first eq. from this last,
and there remains $y=2$

And from eq. 2. $x=14-2y=14-4=10$.

3. Given $\begin{cases} 5y+3z=93 \\ 3y+4z=80 \end{cases}$ to find y and z .

Multiply the first eq. into 3, and the second into

5, and there will arise . . . $15y+9z=279$

And . . . $15y+20z=400$

Subtract the first of these
from the latter, and there
remains $11z=121$

And $z=11$

Whence $y=\frac{93-3z}{5}=\frac{93-33}{5}=\frac{60}{5}=12$.

4. Given $\begin{cases} \frac{y}{2}+\frac{z}{4}=\frac{1}{3} \\ \frac{y}{3}-\frac{z}{5}=\frac{1}{10} \end{cases}$ to find y and z .

Here, dividing the first eq. by 3, and the second
by 2, we have

$$\frac{y}{6}+\frac{z}{12}=\frac{1}{9}$$

$$\text{And . . . } \frac{y}{6}-\frac{z}{10}=\frac{1}{20}$$

Subtract the latter of these } $\frac{x}{12} + \frac{z}{10} = \frac{1}{9} - \frac{1}{20}$
 from the former and .

That is . . . $x + \frac{12z}{10} = \frac{12}{9} - \frac{12}{20}$

And $10x + 12z = \frac{120}{9} - \frac{12}{2}$

That is . . . $22z = \frac{40}{3} - 6$

Or $66z = 40 - 18 = 22$

And . . . $z = \frac{22}{66} = \frac{1}{3}$

Also, $y = \frac{2}{3} - \frac{z}{2}$ (from eq. 1) $= \frac{2}{3} - \frac{1}{6} = \frac{4}{6} - \frac{1}{6}$
 $= \frac{3}{6} = \frac{1}{2}$.

5. Given $\begin{cases} 3x + 8y = 14 \\ 2x - y = 3 \end{cases}$ to find x and y .

Ans. $x=2$. $y=1$.

6. Given $4y - 5z = 2$ and $5y - 4z = 7$ to find y and z .

Ans. $y=3$. $z=2$.

7. Given $\frac{x}{2} + \frac{y}{3} = 8$ and $\frac{x}{3} - \frac{y}{2} = 1$ to find x and y .

Ans. $x=12$. $y=6$.

8. Given $8x + 7y = 259$ and $7x - 8y = 43$ to find x and y .

Ans. $x=21$. $y=13$.

9. Given $\frac{x-1}{2} + \frac{y+1}{3} = 3$ and $\frac{x+1}{3} - \frac{y}{2} = 1$ to find x and y .

Ans. $x=5$. $y=2$.

*PROBLEMS involving two unknown quantities.

1. What two numbers are those whose sum is 20 and difference 12.

The solution by rule 1.

Let the greater number be } x
 called }
 And the less } y
 Then will their sum be . . $x+y=20$
 And their difference . . . $x-y=12$
 From the first of these . . $x=20-y$
 And from the second . . . $x=12+y$
 Wherefore $20-y=12+y$
 Which, by transposition, gives $2y=8$
 And $y=4$
 Whence $x=12+y=12+4=16$

The same by rule 2.

Their sum will be as before $x+y=20$
 And their difference . . . $x-y=12$
 And from the first eq. . . $x=20-y$
 This value substituted for x }
 in the second equation, } $20-y-y=12$
 will give }
 That is $20-2y=12$
 Wherefore $2y=20-12=8$
 And $y=4$
 Consequently $x=20-y=20-4=16$

* Every problem that is truly limited must have as many conditions (or equations) independent of each other as there are unknown quantities to be determined, and no more.

If the conditions of a problem are fewer in number than the unknown quantities, that problem is said to be indeterminate, and admits of several answers; but if there are more conditions than unknown quantities, the problem is either impossible, or the conditions are deducible from each other, and one or more of them superfluous,

The same by rule 3.

Because $x+y=20$

And $x-y=12$

By adding : $2x = 32$ and $x=16$

And by subtracting $2y=8$ and $y=4$

The solution may likewise be performed by making use of one unknown quantity only, according to the four following methods.

First method.

Let the greater number be . . . x

Then will the less be . . . $20-x$

And their difference $-20+2x$

But by the prob. the difference of the numbers is } $2x-20=12$

12, wherefore

And $2x=20+12=32$

Wherefore $x=16$ = the greater

And . . $20-x=20-16=4$ = the less.

Second method.

Let the greater number as } x
before be

Then, because the diff. is } $x-12$
12, the less will be . .

And the sum of these by the } $2x-12=20$
prob. is 20, therefore .

And by transposition and } $x=16$ = the greater
division

And $x-12=16-12=4$ = the less.

Third method.

Let now the less number be . . . y

Then will the greater be . . $20-y$

And their difference $20-2y$

Which, by the prob. is 12, } $20 - 2y = 12$
 therefore }
 And by transposition, &c. . $y = 4$ the less.
 And . . . $20 - y = 20 - 4 = 16$ the greater.

Fourth method.

Let the less number as be- }
 fore be } y
 Then will the greater be . . $y + 12$

And their sum $2y + 12$
 Which is 20 by the prob. } $2y + 12 = 20$
 wherefore }

From whence $y = \frac{20 - 12}{2} = \frac{8}{2} = 4$ the less

And $y + 12 = 4 + 12 = 16$ the greater.

*2. What two numbers are those whose sum is 100,
 and three times the less taken from twice the greater,
 leaves 150 remainder?

Let the greater number be . x
 And the less y
 Then will the first equation } $x + y = 100$
 be }
 And the second $2x - 3y = 150$
 And to solve this by rule }
 1, we have from the first } $x = 100 - y$
 eq. }

And from the second . . . $x = \frac{3y + 150}{2}$

* All the methods in prob. 1 may be applied in this and many of the following problems; the learner should exercise himself well in all those methods, as it will be necessary for him to know which is most suitable to any particular purpose, as they are not all equally so.

Wherefore $\frac{3y+150}{2} = 100 - y$

And $3y + 150 = 200 - 2y$

That is $5y = 50$

And $y = 10$ the less

Consequently . . . $x = 100 - y = 90$ the greater.

3. A bankrupt owed two creditors £140. now the greater debt was to the less as 3 to 2, what sum was due to each?

Let $x =$ the greater debt, and $y =$ the less

Then $x + y = 140$ }
And $x : y :: 3 : 2$ } by the problem

Whence $2x = 3y$ and $x = \frac{3y}{2}$

And substituting this value for x in eq. 1,

we have $\frac{3y}{2} + y = 140$

That is $3y + 2y = 280$

Or $5y = 280$, whence $y = \frac{280}{5} = 56$ the less

And $x = \frac{3y}{2} = \frac{168}{2} = 84$ the greater.

4. A Gentleman being asked the age of his 2 sons; replied that if to the sum of their ages 25 be added, this sum will be double the age of the eldest; but if 8 be taken from the difference of their ages the remainder will be that of the youngest, required the age of each?

Let $x =$ the age of the eldest and $y =$ that of the youngest.

Then $x + y + 25 = 2x$ }
And $x - y - 8 = y$ } by the prob.

PROBLEMS *involving*

Whence $x = y + 25$ from eq. 1.

And $x = 2y + 8$ from eq. 2.

Therefore $y + 25 = 2y + 8$.

And $y = 17$ and $x = (2y + 8) = 42$.

5. Divide £60. between A and B; so that the difference between A's share and 31 may be to the difference of 31 and B's share as 6 to 7.

Let $x =$ A's share and $y =$ B's

Then $x + y = 60$

And $x - 31 : 31 - y :: 6 : 7$ } by the prob.

That is: $(x - 31 \times 7 =) 7x - 217 = (31 - y \times 6 =) 186 - 6y$

Whence $7x = 403 - 6y$ and $x = \frac{403 - 6y}{7}$

Also $y + \frac{403 - 6y}{7} (= x + y) = 60$

Whence $y = 17$ and $x = \left(\frac{403 - 6y}{7}\right) = 43$.

6. A vintner has two sorts of wine, which equally mixed will be worth 2s. 6d. a quart; but if 3 quarts of the best be mixed with 5 quarts of the other, the mixture will be worth but 2s. 4½d. per quart, what is the price of each sort?

Let $x =$ the price of the best sort.

And $y =$ that of the worst

Then will $\frac{x + y}{2}$ (= the price of a quart equally mixed) = 60 (the halfpence in 2s. 6d.)

Also $\frac{3x + 5y}{8}$ (= the pr. of a qt. by the latter condition) = 57 (the halfpence in 2s. 4½d.)

Whence $x = 120 - y$ from eq. 1.

And $x = \frac{456 - 5y}{3}$ from eq. 2.

Therefore $120 - y = \frac{456 - 5y}{3}$

From which $y = 48$ (halfpence) = 2 shillings
And $x (= 120 - y) = 72$ halfpence = 3 shillings.

7. Two men agree to buy a house for £1200. says A to B give me $\frac{2}{3}$ of your money, and I shall be able to purchase it: no, says B, rather give me $\frac{3}{4}$ of your's, and I can purchase it—how much money had each?

Let $x = A$'s money, $y = B$'s

Then will $x + \frac{2y}{3} = 1200$
And . . . $y + \frac{3x}{4} = 1200$ } by the problem

Whence $x = 1200 - \frac{2y}{3}$ from eq. 1

And $x = \frac{4800 - 4y}{3}$ from eq. 2.

Therefore $1200 - \frac{2y}{3} = \frac{4800 - 4y}{3}$

From which $y = 600$

And $x (= 1200 - \frac{2y}{3}) = 800$

8. What fraction is that, to the numerator of which 1 being added the value will be $\frac{1}{3}$, but if 1 be added to the denominator its value will be $\frac{1}{4}$?

Let the fraction be repre- } $\frac{x}{y}$
sented by }

Then will $\frac{x+1}{y} = \frac{1}{3}$

And $\frac{x}{y+1} = \frac{1}{4}$

From eq. 1 $y = 3x + 3$

This substituted for y in eq. 2 gives $\frac{x}{3x+3+1} = \frac{1}{4}$

That is $\frac{x}{3x+4} = \frac{1}{4}$

And by multiplication $4x = 3x + 4$

Therefore $x = 4$ the numerator

And $y = 3x + 3 = 12 + 3 = 15$ the denominator

Therefore the fraction $\frac{x}{y} = \frac{4}{15}$ the answer.

For $\frac{4+1}{15} = \frac{5}{15} = \frac{1}{3}$ }
 And $\frac{4}{15+1} = \frac{4}{16} = \frac{1}{4}$ } the proof.

9. What two numbers are those whose difference is 4, and 5 times the greater is to 6 times the less, as 5 to 4?

Let the numbers be respectively x and y

Then will their difference $x - y = 4$

And $5x : 6y :: 5 : 4$

And multipl. extremes and means, we have $20x = 30y$

But from the first eq. $x = y + 4$

And by substituting $20y + 80 = 30y$

Wherefore $10y = 80$

And $y = 8$

Consequently $x = y + 4 = 12$.

10. What two numbers are those whose sum is 12 and difference 8? *Ans.* 10 and 2.

11. What two numbers are those whose difference is 9, and if 3 times the greater be added to 5 times the less, the sum is 35? *Ans.* 10 and 1.

12. What two numbers are those whose sum is 20, and if from 3 times the greater, 5 times the less be subtracted, the remainder is 4? *Ans.* 13 and 7.

13. There are two numbers such that $\frac{1}{2}$ of the greater added to $\frac{1}{3}$ of the less is 13, and if $\frac{1}{2}$ the less be taken from $\frac{1}{3}$ the greater, the remainder is nothing, what are the numbers? *Ans.* 18 and 12.

14. Says A to B give me 5 shillings of your money, and I shall have twice as much as you will have left; says B rather give me 5 of your shillings, and I shall have 3 times as much as you will have left, what had each? *Ans.* A 11, B 13.

15. Divide £200 between A and B, so that A may have £73. more than B. *Ans.* A's share £136. 10s.
B's . . . £63. 10s.

16. Divide £48. between A and B, so that A's share may be to B's, as 7 to 5. *Ans.* A's share £28. B's £20.

17. A gentleman left £240. between two persons, whose respective shares therein were to be as $\frac{1}{4}$ to $\frac{1}{3}$, how much must each have?

Ans. One £144. the other £96.

18. What 2 numbers are those whose sum is 10, and if half the less be subtracted from twice the greater, the remainder will likewise be 10? *Ans.* 6 and 4.

19. If 2 cows are worth 7 sheep, and 5 cows and 3 sheep together cost 41 pounds, what is the value of each? *Ans.* A cow £7. A sheep £2.

20. In the reign of Edward the sixth a barrel of beer was worth 24 loaves, and a barrel and a half of beer and 12 loaves cost 12 pence, what was the price of each?

Ans. A bar. of beer 6d.

A loaf $\frac{1}{4}$ d.

21. Bought 2 pieces of cloth, one of which measured 7 yards, the other 5; now the first piece, with one yard of the second, cost 68 shillings; and the second, with one yard of the first, cost the same, required the value of each per yard?

Ans. 8s. per yd. the first,

12s. . . the second.

22. In the year 1299, 3 fat oxen and 6 sheep together cost 79 shillings, and the price of an ox exceeded the price of 12 sheep by 10 shillings, required the value of each?

Ans. An ox 24 shillings,

A sheep 1s. 2d.

REDUCTION OF EQUATIONS *containing three unknown quantities.*

RULE.

1. Let x , y , and z be the three unknown quantities.
2. Find the value of x in each of the equations.
3. Put the value of x in the first equation equal to the value of x in the second, and a new equation will be formed, having only two unknown quantities y and z .
4. In like manner put the value of x in the first equation equal to the value of x in the third, and a second new equation will be formed having only two unknown quantities y and z .
5. Find the values of y and z from these two new equations by the preceding rules, from whence the value of x may likewise be obtained.

EXAMPLES.

1. Given $\begin{cases} x+y+z=9 \\ x+2y+3z=16 \\ x+y-2z=3 \end{cases}$ to find the values of x, y and z .

From the first eq. $x=9-y-z$

From the second $x=16-2y-3z$

From the third $x=3-y+2z$

And the first new eq. is $9-y-z=16-2y-3z$

And the second $9-y-z=3-y+2z$

From the first new eq. $y=7-2z$

And from the second $z=2$

Whence $y(=7-2z)=3$

And $x(=9-y-z)=4$.

2. Given $\begin{cases} x+y+z=29 \\ x+2y+3z=62 \\ \frac{x}{2}+\frac{y}{3}+\frac{z}{4}=10 \end{cases}$ to find x, y , and z .

From the first $x=29-y-z$

From the second $x=62-2y-3z$

From the third $x=20-\frac{2y}{3}-\frac{z}{2}$

Whence the first new eq. $29-y-z=62-2y-3z$

And the second $29-y-z=20-\frac{2y}{3}-\frac{z}{2}$

Now from the first new eq. $y=33-2z$

And from the second $y=27-\frac{3z}{2}$

Whence $33-2z=27-\frac{3z}{2}$ and $z=12$

And $y(=33-2z)=9$

Also $x(=29-y-z)=8$.

3. Given $x+y+z=7$, $2x-y-3z=3$, and $5x-3y+5z=19$ to find x , y , and z .

Ans. $x=4$ $y=2$ $z=1$.

*4. Given $x-y-z=5$, $3x+4y+5z=52$, and $5x-4y-3z=32$.

Ans. $x=10$ $y=3$ $z=2$.

AFFECTED QUADRATIC EQUATIONS.

† An affected quadratic equation is that which contains the square of the unknown quantity in one term, and its first power in another.

* Any number of unknown quantities may be exterminated by a method nearly similar to the above, but there are often shorter and more easy methods which will be best learned from practice.

† The term *affected* or *adaffected* was first introduced by Vieta, the great improver of algebra, about the year 1600, to distinguish equations which involve (or are affected with) different powers of the unknown quantity, from others which contain the *biggest power only*, and are therefore called *pure*.

Dr. Hutton calls the former *compound* equations, which name, as Baron Maseres justly observes, is very proper, and less obscure than that of *affected* equations.

This method of resolving affected quadratic equations was known and practised by Lucas de Burgo, who wrote in 1494. And the reasons on which it is founded may be thus explained after the manner of others.

“ The square root of any quantity being either $+$ or $-$, all quadratic equations admit of two solutions, thus, the square root of n^2 is either $+n$ or $-n$, for $+n \times +n$ and $-n \times -n$ is each equal to $+n^2$. But the square root of $-n^2$; that is, $\sqrt{-n^2}$ is impossible, as neither $+n \times +n$ nor $-n \times -n$ will produce $-n^2$.

“ So, in the first form, $x^2+ax=b$ where $x+\frac{1}{2}a$ is found $=\sqrt{b+\frac{1}{4}aa}$, the root may be either $+\sqrt{b+\frac{1}{4}aa}$, or $-\sqrt{b+\frac{1}{4}aa}$, since either of them being multiplied into itself, will produce $b+\frac{1}{4}a^2$. And this ambiguity is expressed by writing the uncertain sign $\pm$ before $\sqrt{b+\frac{1}{4}aa}$, thus, $\pm\sqrt{b+\frac{1}{4}aa}$.

“ In

There are three forms of affected quadratic equations, as follow—

First form $x^2 + ax = b$

Second form $x^2 - ax = b$

Third form $x^2 - ax = -b$.

The value of the unknown quantity in each of these is found by the following.

" In this form the first value of x , viz. $+\sqrt{b+\frac{1}{4}aa}-\frac{1}{2}a$ is always affirmative; for since $\frac{1}{4}a^2+b$ is greater than $\frac{1}{4}a^2$, the greatest square must necessarily have the greatest square root; therefore $\sqrt{b+\frac{1}{4}aa}$ will always be greater than $\sqrt{\frac{1}{4}a^2}$, or its equal $\frac{1}{2}a$; and consequently $+\sqrt{b+\frac{1}{4}aa}-\frac{1}{2}a$ will always be affirmative.

" The second value of x , viz. $-\sqrt{b+\frac{1}{4}aa}$ will always be negative, because it is composed of two negative terms. Therefore, when $x^2+ax=b$ we shall have $x=+\sqrt{b+\frac{1}{4}aa}-\frac{1}{2}a$ for the affirmative value of x , and $x=-\sqrt{b+\frac{1}{4}aa}-\frac{1}{2}a$ for the negative value of x .

" In the second form where $x=+\sqrt{b+\frac{1}{4}aa}+\frac{1}{2}a$, the first value, viz. $x=+\sqrt{b+\frac{1}{4}aa}+\frac{1}{2}a$, is always affirmative, being composed of two affirmative terms. The second value, viz. $x=-\sqrt{b+\frac{1}{4}aa}+\frac{1}{2}a$, will always be negative, for since $b+\frac{1}{4}a^2$ is greater than $\frac{1}{4}a^2$, $\sqrt{b+\frac{1}{4}aa}$ will be greater than $\sqrt{\frac{1}{4}a^2}$, or than its equal $\frac{1}{2}a$; consequently $-\sqrt{b+\frac{1}{4}aa}+\frac{1}{2}a$ is negative; therefore we have $x=+\sqrt{b+\frac{1}{4}aa}+\frac{1}{2}a$, for the affirmative value of x ; and $x=-\sqrt{b+\frac{1}{4}aa}+\frac{1}{2}a$, for the negative value of x ; so that in both the first and second forms, the unknown quantity has always two values, one positive and the other negative.

" In the third form, where $x=+\sqrt{\frac{1}{4}aa-b}+\frac{1}{2}a$, both the values of x will be affirmative, supposing $\frac{1}{4}a^2$ is greater than b . For the first value, viz. $x=+\sqrt{\frac{1}{4}aa-b}+\frac{1}{2}a$ will be affirmative, being composed of two affirmative terms.

" The second value, viz. $x=-\sqrt{\frac{1}{4}aa-b}+\frac{1}{2}a$ is affirmative also; for since $\frac{1}{4}a^2$ is greater than $\frac{1}{4}a^2-b$, $\dots \sqrt{\frac{1}{4}a^2}$ or its equal.

RULE.

1. Transpose all the terms which contain the unknown quantity, to one side of the equation, and the known terms to the other; and let the terms which contain the unknown quantity, be ranged according to their dimensions, as in the above forms.
2. Divide all the terms of the equation by the coefficient of the square of the unknown quantity.
3. Add the square of half the coefficient of the unknown quantity in the second term to both sides of the equation, and that side which contains the unknown quantity will be a complete square.

“equal $\frac{1}{4}a$ is greater than $\sqrt{\frac{1}{4}aa-b}$; consequently $-\sqrt{\frac{1}{4}aa-b} + \frac{1}{2}a$ will always be affirmative. Therefore, when $x^2-ax=-b$, we shall have $x = +\sqrt{\frac{1}{4}aa-b} + \frac{1}{2}a$, and $x = -\sqrt{\frac{1}{4}aa-b} + \frac{1}{2}a$ being both affirmative values of x .

“But in this third form if b be greater than $\frac{1}{4}a^2$, the solution of the proposed problem will be impossible. For since the square of any quantity (whether that quantity be affirmative or negative) is always affirmative, the square root of a negative quantity is impossible, and cannot be assigned. But if b be greater than $\frac{1}{4}a^2$, then $\frac{1}{4}a^2-b$ is negative, therefore $\sqrt{\frac{1}{4}aa-b}$ is impossible or imaginary; consequently, in this case, where $x = \frac{1}{2}a \pm \sqrt{\frac{1}{4}aa-b}$ both the roots, or values of x are impossible or imaginary.”

It may likewise be observed, that though both roots answer the algebraic conditions of a problem, and in the application of algebra to geometry both have a precise and determinate meaning; yet in the solution of a numerical problem the affirmative root only must be considered as the true answer, the negative root being (except in a few particular instances) useless. Also, of the two affirmative roots in the third form, one only will, for the most part, be the true answer, which will be pointed out by the conditions and limitations of the problem, under consideration. When the roots are impossible, it denotes that the problem is absurd, or that its conditions are inconsistent with each other.

4. Extract the square root of both sides of the equation, making that of the known side both + and —.
5. Transpose the known part from the first side, and the value of the unknown quantity will be found.

EXAMPLES.

1. Given $x^2 + 8x = 33$ to find the value of x .

First, $x^2 + 8x + 16 (= 33 + 16) = 49$

Secondly, $\sqrt{x^2 + 8x + 16} = \pm \sqrt{49}$

That is,

Thirdly, $x + 4 = \pm 7$

Fourthly, $x = \pm 7 - 4 = 3$ or -11 .

Explanation.

In the first step the square is completed, that is half 8 (the coefficient of x in the second term) is squared, and its square, viz. 16, added to both sides. Then, in the second step, the square root is extracted from both sides, which produces the third step.

Lastly, by transposing $+4$, the two roots or values of x are obtained.

2. Given $x^2 + 6x = 16$ to find x .

First, $x^2 + 6x + 9 (= 16 + 9) = 25$ by completing the square.

Then $x + 3 = \pm \sqrt{25} = \pm 5$ by extracting the root.

And $x = \pm 5 - 3 = 2$ or -8 by transposition.

3. Given $x^2 - 4x - 4 = 28$ to find x .

First, $x^2 - 4x (= 28 + 4) = 32$ by transposition.

Then $x^2 - 4x + 4 (= 32 + 4) = 36$ by comp. the sq.

And $x - 2 = \pm \sqrt{36} = \pm 6$ by extracting the root.

Consequently, $x = \pm 6 + 2 = 8$ or -4 by trans.

4. Given $3x^2 - 12x + 6 = 21$ to find x .

First, $3x^2 - 12x = 15$ by transposition.

Then $x^2 - 4x = 5$ by division.

Also, $x^2 - 4x + 4 = 9$ by comp. the sq.

And $x - 2 = \pm\sqrt{9} = \pm 3$ by extr. the root.

Therefore $x = \pm 3 + 2 = 5$ or -1 .

5. Given $x^2 + x = 42$ to find x .

First, $x^2 + x + \frac{1}{4} = 42\frac{1}{4}$ by comp. the sq.

Then $x + \frac{1}{2} = \pm\sqrt{42\frac{1}{4}} = \pm\sqrt{42,25} = \pm 6,5$ by extracting the root.

That is $x = \pm 6,5 - \frac{1}{2}$ by transposition.

Or, $x = \pm 6\frac{1}{2} - \frac{1}{2} = 6$ or -7 .

6. Given $\frac{x^2}{2} - \frac{x}{3} + 20\frac{1}{2} = 42\frac{2}{3}$ to find x .

First, $\frac{x^2}{2} - \frac{x}{3} = (42\frac{2}{3} - 20\frac{1}{2}) = 22\frac{1}{6}$ by transp.

And $x^2 - \frac{2x}{3} = 44\frac{1}{3}$ by multiplication.

Also, $x^2 - \frac{2x}{3} + \frac{1}{9} = 44\frac{1}{3} + \frac{1}{9} = 44\frac{4}{9}$ by comp. the sq.

Hence $x - \frac{1}{3} = \pm\sqrt{44\frac{4}{9}} = \pm 6\frac{2}{3}$ by extr. the root.

And $x = \pm 6\frac{2}{3} + \frac{1}{3} = 7$ or $-6\frac{1}{3}$ by transposition.

7. Given $x^2 - 6x = -8$ to find x .

Here $x^2 - 6x + 9 = -8 + 9 = 1$ by comp. the sq.

Then $x - 3 = \pm\sqrt{1} = \pm 1$ by extr. the root.

Also, $x = \pm 1 + 3 = 4$ or 2 by transposition.

8. Given $x^2 + 4x = 32$ to find x . *Ans.* $x = 4$ or -8 .

9. Given $x^2 - 6x + 8 = 80$ to find x .

Ans. $x = 12$ or -6 .

QUADRATIC EQUATIONS. 165

10. Given $x^2 - 4x - 8 = -12$ to find x . *Ans.* $x = 2$.

11. Given $4x^2 + 12x + 8 = 80$ to find x .
Ans. $x = 3$ or -6 .

12. Given $3x^2 - 4x + 5 = 3,68$ to find x .
Ans. $x = 6$ or $,7333+$.

13. Given $\frac{2}{3}x^2 + \frac{3}{4}x = \frac{4}{5}$ to find x .
Ans. $x = ,6689$ or $-1,793$.

RULE 2.

All equations whatever in which there are only two different dimensions of the unknown quantity; if the index of one be exactly double that of the other, may be solved like quadratics, by completing the square, thus—

Having extracted the square root, transpose the quantities which are on the same side with the unknown quantity, and then extract that root from both sides, which the index of the unknown quantity denotes.

EXAMPLES.

1. Given $x^4 + 2x^2 = 24$ to find x .

First, $x^4 + 2x^2 + 1 = 24 + 1 = 25$ by completing the square.

And $x^2 + 1 = \pm\sqrt{25} = \pm 5$ by evolution.

Also, $x^2 = \pm 5 - 1 = 4$ or -6 by transposition.

Whence $x = \pm\sqrt{4} = \pm 2$ or $\pm\sqrt{-6}$ by evolution.

*2. Given $x^6 - 4x^3 = 32$ to find x .

First, $x^6 - 4x^3 + 4 = 32 + 4 = 36$ by completing the square.

And $x^3 - 4 = \sqrt{36} = 6$ by extracting the sq. root.

Also, $x^3 = 4 + 6 = 10$ by transposition.

Therefore, $x = \sqrt[3]{10} = 2,154$ by evolution.

* In this and the following examples one root only is found, the learner may find the rest if he chooses.

3. Given $x+4\sqrt{x}=12$ to find x .

First, $x+4\sqrt{x}+4=12+4=16$ by comp. the sq.

Then $\sqrt{x+2}=\sqrt{16}=4$ by evolution.

And $\sqrt{x}=4-2=2$ by transposition.

Whence $x=2^2=4$ by involution.

4. Given $x^{2n}-x^n=a$ to find x .

First, $x^{2n}-x^n+\frac{1}{4}=a+\frac{1}{4}$ by comp. the sq.

Then $x^n-\frac{1}{2}=\pm\sqrt{a+\frac{1}{4}}$ by extr. the sq. root.

And $x^n=\frac{1}{2}\pm\sqrt{a+\frac{1}{4}}$ by transposition.

Whence $x=\sqrt[n]{\frac{1}{2}\pm\sqrt{a+\frac{1}{4}}}$ by evolution.

5. Given $x^4+2x^2=99$ to find x . *Ans.* $x=3$.

6. Given $x^6-4x^3+4=1$ to find x . *Ans.* $x=1$.

7. Given $2x^4-x^2=496$ to find x . *Ans.* $x=4$.

PROBLEMS *producing Quadratic Equations.*

1. What two numbers are those whose sum is 12 and product 35?

Let x = the greater number.

Then will $12-x$ = the less.

And $x \times (12-x)=35$ by the problem.

That is, $12x-x^2=35$ by multiplication.

And $x^2-12x=-35$ by transposition.

Also, $x^2-12x+36=-35+36=1$ by comp. the square.

And $x-6=\pm\sqrt{1}=\pm 1$ by evolution.

Whence $x=6\pm 1=7$ or 5 by transposition.

And $12-x=12-7$ or $12-5$, viz. 5 or 7 .

Wherefore the numbers are 7 and 5 .

2. The difference of two numbers is 4 and their product 96, what are the numbers?

Let x = the greater.

Then will $x-4$ = the less.

And $x \times (x-4) = x^2 - 4x = 96$ by the prob.

Also, $x^2 - 4x + 4 = 96 + 4 = 100$ by comp. the sq.

Whence $x-2 = 10$ by extr. the root.

And $x = 12$ by transposing -2 .

And $x-4 = 8$.

Wherefore the numbers are 12 and 8.

3. The difference of two numbers is 6, and the sum of their squares 50, what are the numbers?

Let x = the greater.

Then $x-6$ = the less.

And x^2 = square of the greater.

Also, $(x-6)^2 = x^2 - 12x + 36$ = sq. of the less.

Wherefore, $x^2 + x^2 - 12x + 36 = 50$ by the prob.

That is, $2x^2 - 12x = 50 - 36 = 14$ by transp. &c.

And $x^2 - 6x = 7$ by division.

Also, $x^2 - 6x + 9 = 7 + 9 = 16$ by comp. the sq.

Therefore, $x-3 = \sqrt{16} = 4$ by evolution.

Whence $x = 7$ and $x-6 = 1$ = the numbers req.

4. Divide the number 15 into two parts so that their product may be 56.

Let x = the greater.

Then $15-x$ = the less.

And $(15-x) \times x = 56$ by the problem.

That is, $15x - x^2 = 56$ by multiplication.

And $x^2 - 15x = -56$ by transposition.

Also, $x^2 - 15x + 7,5^2 = -56 + 7,5^2$ by comp. the sq.

Therefore, $x - 7,5 = \pm \sqrt{-56 + 7,5^2} = \pm \sqrt{25}$
 $= \pm 5$ by evolution.

And $x = 7,5 \pm 5 = 8$ or $7 =$ the parts required.

For if $x = 8$, then $15 - x = 7$, and if $x = 7$, then
 $15 - x = 8$.

5. What number is that which being tripled, and 8 added to the triple, the sum shall be to the square of the number sought, as 5 to 4?

Let $x =$ the number required.

Then $3x + 8 : x^2 :: 5 : 4$ by the prob.

That is, $12x + 32 = 5x^2$ by mult. extremes and means.

And $5x^2 - 12x = 32$ by transposition.

Also, $x^2 - 2,4x = 6,4$ by division.

And $x^2 - 2,4x + 1,44 = 6,4 + 1,44 = 7,84$ by com.
the square.

Wherefore, $x - 1,2 = \sqrt{7,84} = 2,8$ by evolution.

And $x = 2,8 + 1,2 = 4$ the number required.

6. Two captains distribute each 1200 crowns equally among his men; now A has 40 men more than B; and B's men receive 5 crowns a man more than A's—how many men had each captain?

Let $x =$ B's men.

Then $\frac{1200}{x} =$ the share of each.

Also, $x + 40 =$ A's men.

And $\frac{1200}{x + 40} =$ the share of each.

Wherefore, $\frac{1200}{x} = \frac{1200}{x + 40} + 5$ by the prob.

And $1200x + 48000 = 1200x + 5x^2 + 200$ by multiplication.

That is, $5x^2 + 200x = 48000$ by transposition.

And $x^2 + 40x = 9600$ by division.

Also, $x^2 + 40x + 400 = 10000$ by comp. the sq.

Whence, $x = 80 = \text{B's men.}$

And $x + 40 = 120 = \text{A's men.}$

7. A man bought cloth for 70 shillings, now if there had been 4 yards more, each yard would have cost him 2 shillings less than it did, how many yards were there?

Let $x =$ the number of yards.

Ten $x + 4 =$ the number, with 4 yards more.

Also, $\frac{70}{x} =$ the price per yard according to the first supposition.

And $\frac{70}{x+4} =$ the price per yard by the second.

Whence, $70x + 280 - 2x^2 - 8x = 70x$ by multip.

And $x^2 + 4x = 140$ by transposition, &c.

Wherefore, $x = 10 =$ the number required.

8. One bought a horse and sold him again for £56. and gained as much per cent. as the horse cost—required the price of the horse?

Let $x =$ the price.

Then $56 - x =$ the gain.

And $x : 56 - x :: 100 : \frac{5600 - 100x}{x} =$ the gain per cent.

Whence, $x = \frac{5600 - 100x}{x}$ by the prob.

Also, $x^2 = 5600 - 100x$ by multip.

That is, $x^2 + 100x = 5600$ by transpos.

From which x is found = £40. the sum required.

Q

9. Place 969 men in the form of an oblong or rectangle, so that there may be 40 more in depth than in front?

Let x = the number of men in front.

Then $x+40$ = the number in depth.

And $(x+40) \times x$

That is, $x^2 + 40x = 969$ } by the prob.

And completing the square, &c. we have

$$x=17 \text{ and } x+40=57.$$

10. A company at a tavern had £7. 4s. to pay, but before the reckoning was settled two of them sneaked off, which obliged the rest to pay a shilling a piece more than they should have done, how many persons were there?

Let x = the number of persons.

Then $144 = £7. 4s.$ reduced to shillings.

Also, $\frac{144}{x} =$ what each *should have paid*.

And $\frac{144}{x-2} =$ what each *did pay*.

Wherefore, $\frac{144}{x-2} = \frac{144}{x} + 1$ by the problem.

Which reduced gives $x=18$ the number required.

11. Bought 120lb. of pepper, and as many of ginger, and received for a crown one pound more of ginger than of pepper—now the pepper cost 6 crowns more than the ginger, how many pounds were there of each?

Let x = the lbs. of pepper for a crown.

Then $x+1$ = ditto of ginger.

Also, $\frac{120}{x} =$ the price of the pepper.

And $\frac{120}{x+1} =$ ditto of the ginger.

Whence, $\frac{120}{x+1} + 6 = \frac{120}{x}$ by the problem.

Which reduced gives $x=4$ lb. of pepper, and
 $x+1=5$ lb. of ginger.

12. One buys cloth for £33. 15s. which he sells again at 48 shillings per piece, and gained as much as a piece cost—required the number of pieces?

Let x = the number of pieces.

Then 675 = the number of shillings in £33. 15s.

And $\frac{675}{x}$ = what each piece cost.

Also, $48x$ = what he sold the whole for.

Wherefore, $48x - 675 = \frac{675}{x}$ by the problem.

Whence $x=15$.

13. A grazier bought as many sheep as cost him £60. and after reserving 15, he sold the remainder for £54. and gained 2 shillings a head by them, how many sheep did he buy?

Let x = the number of sheep.

Then 1200 = the number of shill. in £60.

And 1080 = ditto in £54.

Also, $\frac{1200}{x}$ = what each sheep cost.

And $\frac{1080}{x-15}$ = what each sold for.

Wherefore, $\frac{1080}{x-15} - \frac{1200}{x} = 2$ by the prob.

Which reduced gives $x=75$ the number required.

14. Two partners gained £140. A's money was 3 months in trade, and his gain was £60. less than

his stock; and B's money, which was £50. more than A's, was in trade 5 months, what was A's stock?

Let x = A's stock.

Then $x+50$ = B's stock.

And $x-60$ = A's gain.

Also, $140-(x-60)=200-x$ = B's gain.

Now A's stock $\times$ its time : A's gain :: B's stock $\times$ its time : B's gain.

That is,

$$3x : x-60 :: (x+50) \times 5 : 200-x.$$

Or, $3x : x-60 :: 5x+250 : 200-x$ by mul.

$$\text{From which } x^2 - \frac{650}{8}x = \frac{15000}{8}, \text{ this resolved,}$$

Gives $x=100$ = A's stock.

15. Given the sum of two numbers 12, and the sum of their squares 90, to find those numbers.

Ans. 9 and 3.

16. Divide 36 into two parts, so that their product may be 288.

Ans. 24 and 12.

17. A person bought a number of oxen for £80. now had he bought 4 more for the same money he would have paid £1. less for each, how many did he buy?

Ans. 16.

18. A member of Parliament ordered £7. 4s. to be distributed among the poor voters; when they were assembled, and the steward ready to distribute the money, there comes in unexpectedly two more claimants, by which those at first assembled received a shilling a piece less than they otherwise would have done, what was the number at first?

Ans. 16.

19. Towards the expence of building a ship, A paid £2000. more than B, and £9000. more than C;

also, A's payment was equal to the sum of the squares of B's and C's; how much did each contribute.

Ans. A £17000. B £15000. C £8000.

20. In a rectangular pavement consisting of 1000 square stones, there are 15 more in length than in breadth—how many stones are there in length and breadth?

Ans. 40 and 25.

21. Bought two remnants of cloth, one of which was 6 yards longer than the other, for 68 shillings; now each of them cost as many shillings a yard as there were yards therein, what was the length of each?

Ans. The greater 8 yards, the less 2 yards.

S U R D S.

*Surds are quantities under the radical sign which have no exact roots; they are likewise named *Irrational quantities*, to distinguish them from *rational* ones, the exact roots of which can be found.

Thus, $\sqrt{2}$, $\sqrt[3]{3}$, $\sqrt[4]{8}$, &c. are surds having no exact roots.

* This is the true definition of surds; that given in page 5 is a little different, being only intended to convey the most general idea by including all radical quantities under the general name of surds.

The reduction of surds consists in changing them from one form to another without altering their values, for the greater ease in adding, subtracting, multiplying, dividing, or approximating to their values. The reasons on which the several operations are founded will be sufficiently plain to those who are well acquainted with Fractions, Involution, and Evolution; and the learner will do well to exercise his skill in endeavouring to account for them as he goes on; it will likewise now be proper for him to turn back, and re-examine those matters in the former part of the book which he did not perfectly understand, and to work out all the examples which were omitted in the first reading.

And $\sqrt{4}$, $\sqrt[3]{27}$, $\sqrt[4]{256}$, &c. are rational quantities whose exact roots are 2, 3 and 4 respectively.

1. To reduce a rational quantity to the form of a surd.

RULE.

Involve the quantity to a power equivalent to that denoted by the index of the surd, and over this power place the radical sign, and it will be of the form required.

EXAMPLES.

1. Reduce 5 to the form of the square root.

First, $5 \times 5 = 5^2 = 25$.

Whence $\sqrt{25}$ is the answer.

2. Reduce 3 to the form of the cube root.

Here $\sqrt[3]{3 \times 3 \times 3} = \sqrt[3]{27}$ the answer.

3. Reduce x to the form of the 4th root.

Here $\sqrt[4]{x \times x \times x \times x} = \sqrt[4]{x^4}$ the answer.

4. Reduce $x^{\frac{1}{2}}$ to the form of the square root.

Here $\sqrt{x^{\frac{1}{2}} \times x^{\frac{1}{2}}} = \sqrt{x^1} = \sqrt{x}$ the answer.

5. Reduce $4ay$ to the form of the 5th root.

Here $\sqrt[5]{4ay \times 4ay \times 4ay \times 4ay \times 4ay} = \sqrt[5]{1024a^5y^5}$ the answer.

6. Reduce 4 to the form of the square root, and $\frac{1}{8}x$ to the form of the cube root.

Ans. $\sqrt{16}$ and $\sqrt[3]{\frac{1}{8}x^3}$.

7. Reduce $a+z$ to the form of the square root.

$$\text{Ans. } (a^2 + 2az + z^2)^{\frac{1}{2}}.$$

2. To reduce quantities to other equivalent ones having a common index.

RULE.

Reduce the indices to fractions having a common denominator, and place the new numerators each over its respective quantity for a new index. Place 1 over the common denominator, and then write this fraction as an index over the given quantities with their new indices.

EXAMPLES.

1. Reduce $7^{\frac{3}{4}}$ and $8^{\frac{1}{2}}$ to equivalent quantities having a common index.

$$\text{First, } \begin{array}{l} 3 \times 2 = 6 \\ 1 \times 4 = 4 \end{array} \left. \vphantom{\begin{array}{l} 3 \times 2 = 6 \\ 1 \times 4 = 4 \end{array}} \right\} \text{ new numerators.}$$

$$\text{And } 4 \times 2 = 8 \text{ common denominator.}$$

Therefore $7^{\frac{6}{8}}$ and $8^{\frac{4}{8}}$ are the quantities required.

$$\text{But } 7^6 = 117649 \text{ and } 8^4 = 4096.$$

Therefore $117649^{\frac{1}{8}}$ and $4096^{\frac{1}{8}}$ are the answer.

2. Reduce $x^{\frac{1}{2}}$ and $x^{\frac{1}{3}}$ to a common index.

$$\begin{array}{l} 1 \times 5 = 5 \\ 1 \times 4 = 4 \end{array} \left. \vphantom{\begin{array}{l} 1 \times 5 = 5 \\ 1 \times 4 = 4 \end{array}} \right\} \text{ new num.}$$

$$4 \times 5 = 20 = \text{com. denom.}$$

Therefore, $x^{\frac{5}{20}}$ and $x^{\frac{4}{20}}$ the answer.

3. Reduce $a^{\frac{1}{n}}$ and $x^{\frac{1}{m}}$ to a common index.

$$\begin{array}{l} 1 \times m = m \\ 1 \times n = n \end{array} \left. \vphantom{\begin{array}{l} 1 \times m = m \\ 1 \times n = n \end{array}} \right\} \text{new num.}$$

$$n \times m = mn \text{ com denom.}$$

Therefore, $a^{\frac{1}{mn}}$ and $x^{\frac{1}{nn}}$ the answer.

4. Reduce $\sqrt[n]{a+x}$ and $\sqrt[n]{a-x}$ to a common index.

$$\begin{array}{l} 1 \times 3 = 3 \\ 1 \times 2 = 2 \end{array} \left. \vphantom{\begin{array}{l} 1 \times 3 = 3 \\ 1 \times 2 = 2 \end{array}} \right\} \text{new num.}$$

$$2 \times 3 = 6 \text{ com. denom.}$$

Therefore, $(a+x)^{\frac{1}{6}}$ and $(a-x)^{\frac{1}{6}}$ being actually involved are $\sqrt[6]{a^3+3a^2x+3ax^2+x^3}$ and $\sqrt[6]{a^3-3a^2x+3ax^2-x^3}$ the answer.

5. Reduce $3^{\frac{1}{2}}$ and $4^{\frac{1}{3}}$ to a common index.

$$\text{Ans. } 27^{\frac{1}{6}} \text{ and } 16^{\frac{1}{6}}.$$

6. Reduce $a^{\frac{3}{4}}$ and $b^{\frac{5}{6}}$ to a common index.

$$\text{Ans. } a^{\frac{9}{24}} \text{ and } b^{\frac{20}{24}}.$$

7. Reduce $\sqrt[n]{x+y}$ and $\sqrt[n]{x-y}$ to a common index.

$$\text{Ans. } \sqrt[n]{x^3+2xy+y^2} \text{ and } \sqrt[n]{x^3-3x^2y+3xy^2-y^3}.$$

3. To reduce quantities to equivalent ones having a given index.

RULE.

Divide the indices of the quantities by the given index, and place the quotients over the said quantities for new indices.

Over the quantities, with their new indices, place the given index, and the result will be the equivalent quantities required.

EXAMPLES.

1. Reduce $12^{\frac{1}{2}}$ to an equivalent quantity whose index is $\frac{1}{3}$.

First, $\frac{1}{2} \div \frac{1}{3} = \frac{1}{2} \times \frac{3}{1} = \frac{3}{2}$ the new index.

Therefore, $12^{\frac{3}{2}}$ or its equal $1728^{\frac{1}{3}}$ is the answer.

2. Reduce $4^{\frac{1}{2}}$ and $5^{\frac{1}{3}}$ to equivalent quantities having the given index $\frac{1}{6}$.

$\frac{1}{2} \div \frac{1}{6} = \frac{1}{2} \times \frac{6}{1} = 3$ the first index.

$\frac{1}{3} \div \frac{1}{6} = \frac{1}{3} \times \frac{6}{1} = 2$ the second index.

Therefore, $(4^3)^{\frac{1}{6}}$ and $(5^2)^{\frac{1}{6}}$ are the quantities required.

3. Reduce $a^{\frac{1}{m}}$ and $b^{\frac{1}{n}}$ to equivalent quantities having the common index $\frac{r}{s}$.

$\frac{1}{m} \div \frac{r}{s} = \frac{1}{m} \times \frac{s}{r} = \frac{s}{mr}$ the first index.

$\frac{1}{n} \div \frac{r}{s} = \frac{1}{n} \times \frac{s}{r} = \frac{s}{nr}$ the second index.

Therefore, $(a^{\frac{s}{mr}})^{\frac{r}{s}}$ and $(b^{\frac{s}{nr}})^{\frac{r}{s}}$ the answer.

4. Reduce $2^{\frac{1}{2}}$ and $3^{\frac{1}{3}}$ to the common index $\frac{1}{6}$.

Ans. $4^{\frac{1}{6}}$ and $27^{\frac{1}{6}}$.

5. Reduce $x^{\frac{1}{2}}$ and $y^{\frac{1}{3}}$ to the common index $\frac{1}{6}$.

Ans. $(x^4)^{\frac{1}{6}}$ and $(y^2)^{\frac{1}{6}}$.

*4. To reduce surds to their simplest terms.

RULE.

Divide the given surd by the greatest power that will divide it without a remainder—and place the said power and quotient connected by the sign $\times$ of multiplication, under the radical sign —.

Extract the root of the forementioned power, and place its root before the said quotient, with the proper radical sign between them.

EXAMPLES.

1. Reduce $\sqrt{18}$ to its most simple terms.

$$\sqrt{18} = \sqrt{9 \times 2} = \sqrt{9} \times \sqrt{2} = 3 \times \sqrt{2} = 3\sqrt{2} \text{ the ans.}$$

Explanation.

The greatest square number which will divide 18 is 9, and the quotient is 2. These are placed under the radical sign, and then the root of 9 is extracted, which is 3, and this annexed to the surd part is $3\sqrt{2}$ as above.

2. Reduce $\sqrt{24}$ to its simplest terms.

$$\sqrt{24} = \sqrt{4 \times 6} = 2\sqrt{6} \text{ the answer.}$$

* When the given surd contains no exact power, it is already in its simplest terms.

3. Reduce $\sqrt[3]{54a^4}$ to its simplest terms.

$$\sqrt[3]{54a^4} = \sqrt[3]{27a^3 \times 2a} = 3a\sqrt[3]{2a} \text{ the answer.}$$

4. Reduce $\sqrt{12x^3y - 16x^3}$ to its most simple terms.

$$\sqrt{12x^3y - 16x^3} = \sqrt{4x^2 \times (3xy - 4x^3)} = 2x\sqrt{3y - 4x^3} \text{ anf.}$$

5. Reduce $\sqrt{50}$ to its most simple terms.

$$\text{Ans. } 5\sqrt{2}.$$

6. Reduce $\sqrt[3]{16a^4}$ to its simplest terms.

$$\text{Ans. } 2a\sqrt[3]{2a}.$$

7. Reduce $\sqrt[3]{128a^3x}$ and $\sqrt[3]{2a^4x}$ to their simplest terms.

$$\text{Ans. } 4a\sqrt[3]{2x} \text{ and } a\sqrt[3]{2x}.$$

8. Reduce $(64a^5x^2 - 32a^6x)^{\frac{1}{3}}$ to its simplest terms.

$$\text{Ans. } 2a\sqrt[3]{2x^2 - ax}.$$

* 5. The quantity under the radical sign being a fraction, to reduce it to a whole number.

RULE.

Multiply both terms of the fraction under the radical sign, into that power of its denominator whose index is 1 less than the index of the surd.

Then the denominator of the resulting fraction may be taken away, provided you divide the quantity without the radical sign by its root, so shall the surd part be a whole number.

* The value of an integral surd is more easily estimated than that of a fraction alone.

EXAMPLES.

1. Given $\frac{3}{4}\sqrt{\frac{4}{5}}$ to reduce the radical part to a whole number.

First, both terms of the fraction $\sqrt{\frac{4}{5}}$ being multiplied into 5, the denominator will be 25, a complete square, that is the fraction $\sqrt{\frac{4}{5}} = \sqrt{\frac{4 \times 5}{5 \times 5}} = \sqrt{\frac{20}{25}}$.

Now taking away this denominator and dividing $\frac{3}{4}$, (the rational part) by 5 its root, the result is $\frac{3}{4 \times 5}\sqrt{20}$ or $\frac{3}{20}\sqrt{20}$, this reduced to its simplest terms will be $\frac{3}{20}\sqrt{4 \times 5} = \frac{3 \times 2}{20}\sqrt{5} = \frac{6}{20}\sqrt{5} = \frac{3}{10}\sqrt{5}$ the answer.

2. Reduce the radical part of $\frac{2}{3}\sqrt{\frac{1}{2}}$ to a whole number.

First, $\sqrt{\frac{1 \times 2}{2 \times 2}} = \sqrt{\frac{2}{4}}$, whose denominator 4 is an exact square.

$$\text{Then } \frac{2}{3} \div 2 = \frac{1}{3}.$$

Therefore, $\frac{1}{3}\sqrt{2}$ the answer.

3. Reduce the radical part of $3\sqrt[3]{\frac{4}{7}}$ to a whole number.

First, $\sqrt[3]{\frac{4}{7}} = \sqrt[3]{\frac{4 \times 49}{7 \times 49}} = \sqrt[3]{\frac{196}{343}}$ whose denom. is the cube of 7.

Then $3 \div 7 = \frac{3}{7}$ the rational part.

Whence, $\frac{3}{7} \sqrt[3]{196}$ the answer.

4. Reduce $\sqrt[4]{\frac{1}{3}}$ to an equal quantity whose surd part shall be a whole number.

$$\sqrt[4]{\frac{1}{3}} = \sqrt[4]{\frac{1 \times 3 \times 3 \times 3}{3 \times 3 \times 3 \times 3}} = \sqrt[4]{\frac{1 \times 27}{3 \times 27}} = \sqrt[4]{\frac{27}{81}}.$$

Whence, $\frac{1}{3} \sqrt[4]{27}$ the answer.

5. Reduce $\sqrt{\frac{50}{147}}$ to its simplest terms, having its radical part a whole number. Ans. $\frac{5}{21} \sqrt{6}$.

6. Given $\sqrt[3]{\frac{32}{250}}$ and $4\sqrt{\frac{3}{7}}$ to find their simplest terms, having their surd parts whole numbers.

$$\text{Ans. } \frac{2}{5} \sqrt[3]{2} \text{ and } \frac{4}{7} \sqrt{21}.$$

6. To add surds together.

Reduce the quantities to a common index, the fractions to a common denominator, and the surds to their simplest terms. If the surd part be the same in all the quantities, add the rational parts (or coefficients)

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cients)

cients) together, and annex their sum to the common surd. But if the surd part be not the same, the quantities can be added by means of their signs only.

EXAMPLES.

1. Required the sum of $\sqrt{8}$, $\sqrt{18}$, and $\sqrt{32}$.

These reduced to their simplest terms are

$$\begin{aligned}\sqrt{8} &= \sqrt{4 \times 2} = 2\sqrt{2} \\ \sqrt{18} &= \sqrt{9 \times 2} = 3\sqrt{2} \\ \sqrt{32} &= \sqrt{16 \times 2} = 4\sqrt{2}\end{aligned}$$

And their sum $9\sqrt{2}$ the ans. required.

Explanation.

Having reduced the surds to their simplest terms, the coefficients 2, 3, and 4, are added together, and their sum 9 is prefixed to the common surd $\sqrt{2}$, making the sum $9\sqrt{2}$.

2. Add $\sqrt[3]{24}$ and $\sqrt[3]{192}$ together.

These reduced to their simplest terms are

$$\begin{aligned}\sqrt[3]{24} &= \sqrt[3]{8 \times 3} = 2\sqrt[3]{3} \\ \sqrt[3]{192} &= \sqrt[3]{64 \times 3} = 4\sqrt[3]{3}\end{aligned}$$

These added together give $6\sqrt[3]{3}$ the sum required.

3. Add $\sqrt{4a^2x}$ and $\sqrt{16a^2x}$ together.

$$\begin{aligned}\sqrt{4a^2x} &= 2a\sqrt{x} \\ \sqrt{16a^2x} &= 4a\sqrt{x}\end{aligned}$$

Their sum $6a\sqrt{x}$ the answer.

4. Add $\sqrt[3]{8a^3x^2}$ and $\sqrt[3]{16a^3x^2}$ together.

$$\sqrt[3]{8a^3x^2} = 2a\sqrt[3]{x^2} \text{ and } \sqrt[3]{16a^3x^2} = 2a\sqrt[3]{2x^2}.$$

These surds being unlike their sum can be denoted only by connecting them by their proper signs, therefore $2a\sqrt[3]{x^2} + 2a\sqrt[3]{2x^2}$ is the sum required.

5. Add $\sqrt{\frac{1}{6}}$ and $\sqrt{\frac{8}{27}}$ together.

These reduced first to a common denominator, and then to their simplest terms, become

$$\sqrt{\frac{1}{6}} = \sqrt{\frac{27}{162}} = \sqrt{\frac{9 \times 3}{81 \times 2}} = \frac{3}{9} \sqrt{\frac{3}{2}}$$

$$\sqrt{\frac{8}{27}} = \sqrt{\frac{48}{162}} = \sqrt{\frac{16 \times 3}{81 \times 2}} = \frac{4}{9} \sqrt{\frac{3}{2}}$$

These added together give $\frac{7}{9} \sqrt{\frac{3}{2}} = \frac{7}{18} \sqrt{6}$ the sum required.

7. To subtract surd quantities.

RULE.

Prepare the quantities, as in addition, and if the surd parts are alike, subtract their coefficients, annexing the remainder to the common surd. But if the surd parts are not alike, change the sign of the quantity to be subtracted, and then connect it with the other quantity.

EXAMPLES.

1. Required the difference of
- $\sqrt{24}$
- and
- $\sqrt{6}$
- .

First, reducing $\sqrt{24}$ to its simplest terms, we have

$$\sqrt{24} = \sqrt{4 \times 6} = 2\sqrt{6}$$

*Then subtracting 1 from 2 the remainder is 1 (understood).**Therefore, $\sqrt{24} - \sqrt{6} = 2\sqrt{6} - \sqrt{6} = 1\sqrt{6}$ or $\sqrt{6}$ the answer.*

2. Required the difference of
- $\sqrt[3]{128}$
- and
- $\sqrt[3]{16}$
- .

These reduced to their simplest terms are

$$\sqrt[3]{128} = \sqrt[3]{64 \times 2} = 4\sqrt[3]{2}$$

$$\sqrt[3]{16} = \sqrt[3]{8 \times 2} = 2\sqrt[3]{2}$$

Therefore $4\sqrt[3]{2} - 2\sqrt[3]{2} = 2\sqrt[3]{2}$ the difference required.

3. What is the difference of
- $\sqrt[4]{32a^5x}$
- and
- $\sqrt[4]{2a^5x}$
- .

$$\sqrt[4]{32a^5x} = \sqrt[4]{16a^4 \times 2ax} = 2a\sqrt[4]{2ax}$$

$$\sqrt[4]{2a^5x} = \sqrt[4]{a^4 \times 2ax} = a\sqrt[4]{2ax}$$

Whence, $2a\sqrt[4]{2ax} - a\sqrt[4]{2ax} = a\sqrt[4]{2ax}$ the answer req.

4. Required the difference of
- $\frac{2}{3}$
- and
- $\sqrt{\frac{27}{50}}$
- .

These reduced to a common denominator become

$$\sqrt{\frac{2}{3}} = \sqrt{\frac{100}{150}} \text{ and } \sqrt{\frac{27}{50}} = \sqrt{\frac{81}{150}}$$

Now, reducing these to their simplest terms, we have

$$\sqrt{\frac{100}{150}} = \sqrt{\frac{100 \times 1}{25 \times 6}} = \frac{10}{5} \sqrt{\frac{1}{6}}$$

$$\sqrt{\frac{81}{150}} = \sqrt{\frac{81 \times 1}{25 \times 6}} = \frac{9}{5} \sqrt{\frac{1}{6}}$$

Therefore, $\frac{10}{5}\sqrt{\frac{1}{6}} - \frac{9}{5}\sqrt{\frac{1}{6}} = \frac{1}{5}\sqrt{\frac{1}{6}}$

Which, by making the surd part a whole number, becomes

$\frac{1}{5}\sqrt{\frac{1 \times 6}{6 \times 6}} = \frac{1}{5}\sqrt{\frac{6}{36}}\sqrt{\frac{1}{5 \times 6}}\sqrt{6} = \frac{1}{30}\sqrt{6}$ the answer.

5. From $\sqrt{128}$ take $\sqrt{8}$. *Ans.* $6\sqrt{2}$.

6. From $2\sqrt{50}$ take $\sqrt{18}$ *Ans.* $7\sqrt{2}$.

7. From $\sqrt[3]{192}$ take $\sqrt[3]{24}$. *Ans.* $2\sqrt[3]{3}$.

8. From $\sqrt[3]{\frac{2}{3}}$ take $\sqrt[3]{\frac{9}{32}}$. *Ans.* $\frac{1}{12}\sqrt[3]{18}$.

9. From $\sqrt{6a^2}$ take $\sqrt{2a^2}$. *Ans.* $a\sqrt{6} - a\sqrt{2}$.

10. From $\sqrt[4]{162a^3x^2}$ take $\sqrt[4]{2a^3x^2}$. *Ans.* $2a\sqrt[4]{2ax^2}$.

11. From $3\sqrt[5]{64a^5x - 32a^6x^2}$ take $\sqrt[5]{64a^5x - 32a^6x^2}$.
Ans. $4a\sqrt[5]{2x - ax^2}$.

8. To multiply surd quantities together.

RULE.

Reduce the surds to the same index, then multiply the coefficients (or rational parts) together for the rational part of the product, and the surd parts together for the surd part.

Annex the former product to the latter, and reduce the surd to its simplest terms.

EXAMPLES.

1. Multiply $4\sqrt{3}$ into $5\sqrt{6}$.

The prod. of the rational part is $4 \times 5 = 20$.

And of the surd parts $\sqrt{3} \times \sqrt{6} = \sqrt{18}$.

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These connected give $20\sqrt{18}$, which reduced to its simplest terms becomes $20\sqrt{9} \times 2 = 20 \times 3\sqrt{2} = 60\sqrt{2}$ the product required.

2. Multiply $4\sqrt{8}$ and $5\sqrt{3}$ together.

$4\sqrt{8} \times 5\sqrt{3} = 20\sqrt{24} = 20\sqrt{4 \times 6} = 40\sqrt{6}$ the product.

3. Multiply $\sqrt{4ax}$ into $\sqrt{3ab}$.

$\sqrt{4ax} \times \sqrt{3ab} = \sqrt{12a^2bx} = \sqrt{4a^2 \times 3bx} = 2a\sqrt{3bx}$.

4. Multiply $\sqrt[3]{7a}$ into $\sqrt[3]{5ab^2}$.

$\sqrt[3]{7a} \times \sqrt[3]{5ab^2} = \sqrt[3]{35a^2b^2}$ the answer.

5. Multiply $\sqrt{\frac{4}{7}}$ into $\sqrt{\frac{5}{8}}$.

$\sqrt{\frac{4}{7}} \times \sqrt{\frac{5}{8}} = \sqrt{\frac{20}{56}} = \sqrt{\frac{4 \times 5}{4 \times 14}} = \sqrt{\frac{5}{14}} = \sqrt{\frac{5 \times 14}{14 \times 14}}$
 $= \frac{5}{14}\sqrt{14}$ the answer.

6. Multiply $x + \sqrt{y}$ into $x - \sqrt{y}$, and $3 + \sqrt[3]{2}$ into $4 - \sqrt[3]{3}$.

$$\begin{array}{r} x + \sqrt{y} \\ x - \sqrt{y} \\ \hline \end{array}$$

$$\begin{array}{r} x^2 + x\sqrt{y} \\ -x\sqrt{y} - y \\ \hline \end{array}$$

$$\begin{array}{r} x^2 \qquad -y \text{ prod.} \\ \hline \end{array}$$

$$\begin{array}{r} 3 + \sqrt[3]{2} \\ 4 - \sqrt[3]{3} \\ \hline \end{array}$$

$$\begin{array}{r} 12 + 4\sqrt[3]{2} \\ -3\sqrt[3]{3} - \sqrt[3]{6} \\ \hline \end{array}$$

$$\begin{array}{r} 12 + 4\sqrt[3]{2} - 3\sqrt[3]{3} - \sqrt[3]{6} \text{ prod.} \\ \hline \end{array}$$

7. Multiply $(x+y)^{\frac{1}{2}}$ into $(x+y)^{\frac{1}{2}}$.

These reduced to a common index become

$$(x+y)^{\frac{1}{2}} = (x+y)^{\frac{3}{6}} = \overline{(x+y)^3}^{\frac{1}{6}}$$

$$(x+y)^{\frac{1}{2}} = (x+y)^{\frac{3}{6}} = \overline{(x+y)^3}^{\frac{1}{6}}$$

Wherefore, $\overline{(x+y)^3}^{\frac{1}{6}} \times \overline{(x+y)^3}^{\frac{1}{6}} = \overline{(x+y)^6}^{\frac{1}{6}} =$

$(x^3 + 5x^2y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^3)^{\frac{1}{6}}$ the product required.

8. Multiply $5\sqrt{5}$ into $3\sqrt{8}$. *Ans.* $30\sqrt{10}$.

9. Multiply $\frac{2}{3}\sqrt{18}$ into $\frac{5}{4}\sqrt{4}$. *Ans.* $10\frac{1}{2}\sqrt{9}$.

10. Multiply $\frac{2}{3}\sqrt{\frac{1}{2}}$ into $\frac{1}{2}\sqrt{\frac{5}{6}}$. *Ans.* $\frac{1}{18}\sqrt{15}$.

11. Multiply $x^{\frac{1}{2}}$ into $x^{\frac{1}{2}}$. *Ans.* x .

12. Multiply $2^{\frac{1}{2}}$ into $3^{\frac{1}{2}}$. *Ans.* $(648)^{\frac{1}{2}}$.

9. To divide one surd by another.

RULE.

Reduce the surds to the same index, and divide the rational parts by the rational; and the surd by the surd; the former quotient annexed to the latter will be the quotient required, which may be reduced to its simplest terms as before.

EXAMPLES.

1. Let
- $12\sqrt{15}$
- be divided by
- $4\sqrt{3}$
- .

First, $\frac{12}{4} = 3$ for the rational part.

And $\frac{\sqrt{15}}{\sqrt{3}} = \sqrt{5}$ for the surd part.

Wherefore, $3\sqrt{5}$ is the quotient required.

2. Divide
- $8\sqrt[3]{48}$
- by
- $2\sqrt[3]{2}$
- .

Ans. $\frac{8\sqrt[3]{48}}{2\sqrt[3]{2}} = 4\sqrt[3]{24} = 4\sqrt[3]{8 \times 3} = 8\sqrt[3]{3}$ the quotient.

3. Divide
- $\frac{3}{4}\sqrt{\frac{5}{6}}$
- by
- $\frac{1}{2}\sqrt{\frac{1}{3}}$
- .

$\frac{3}{4} \div \frac{1}{2} = \frac{3}{2}$ the rational part.

$\sqrt{\frac{5}{6}} \div \sqrt{\frac{1}{3}} = \sqrt{\frac{5}{2}}$ the surd part.

Wherefore, $\frac{3}{2}\sqrt{\frac{5}{2}} = \frac{3}{2}\sqrt{\frac{5}{2} \times \frac{2}{2}} = \frac{3}{2}\sqrt{\frac{10}{4}} = \frac{3}{4}\sqrt{10}$ the quotient.

4. Divide
- $x^2 - y$
- by
- $x + \sqrt{y}$
- and
- $16 - \sqrt[3]{9}$
- by
- $4 - \sqrt[3]{3}$
- .

$$\begin{array}{r}
 x + \sqrt{y} \overline{) x^2 - y} \qquad \qquad \qquad 4 - \sqrt[3]{3} \overline{) 16 - \sqrt[3]{9}} \\
 \underline{x^2 + x\sqrt{y}} \qquad \qquad \qquad \underline{16 - 4\sqrt[3]{3}} \\
 -x\sqrt{y} - y \qquad \qquad \qquad 4\sqrt[3]{3} - \sqrt[3]{9} \\
 \underline{-x\sqrt{y} - y} \qquad \qquad \qquad \underline{4\sqrt[3]{3} - \sqrt[3]{9}}
 \end{array}$$

5. Divide $8\sqrt{108}$ by $2\sqrt{6}$. *Ans.* $12\sqrt{2}$.
 6. Divide $3\sqrt{8}$ by $2\sqrt{2}$. *Ans.* 3.
 7. Divide $10\sqrt[3]{108}$ by $5\sqrt[3]{4}$. *Ans.* 6.
 8. Divide $4\sqrt[3]{1000}$ by $2\sqrt[3]{4}$. *Ans.* $10\sqrt[3]{2}$.
 9. Divide $\sqrt{\frac{10}{11}}$ by $\sqrt{\frac{2}{3}}$. *Ans.* $\frac{1}{11}\sqrt{165}$.
 10. Divide $\sqrt[3]{\frac{3ax}{4x}}$ by $\sqrt[3]{\frac{ay}{5x}}$. *Ans.* $\frac{1}{2y}\sqrt[3]{30y^2x}$.
 11. Divide $x^2 - x^2\sqrt{x}$ by $x - x\sqrt{x}$. *Ans.* $x + x\sqrt{x} + x^2$.

10. *Involution of surds.*

RULE.

Multiply the index of the surd into the index of the power to which it is to be raised, and to the result annex the power of the rational parts, and it will give the power required.

EXAMPLES.

1. Involve $4\sqrt[3]{x}$ to the third power.

$$4 \times 4 \times 4 = 64 \text{ the rational part.}$$

$$\sqrt[3]{x} = x^{\frac{1}{3}}$$

Therefore, $x^{\frac{1}{3}} \times 3 = x^{\frac{3}{3}} = x$ the 3d power of the surd part.

Whence $64x$ = the answer required.

2. Involve $\frac{2}{3}\sqrt{x}$ to the square.

$$\frac{2}{3} \times \frac{2}{3} = \frac{4}{9} \text{ the rational part.}$$

$$x^{\frac{1}{5}} \times 2 = x^{\frac{2}{5}} = x^2 \sqrt[5]{} \text{ the surd part.}$$

Wherefore, $\frac{4}{9} \sqrt[5]{x^2}$ is the answer.

3. It is required to find the square of $\frac{2}{3} \sqrt{\frac{5}{7}}$.

$$\frac{2}{3} \times \frac{2}{3} = \frac{4}{9} \text{ and } \sqrt{\frac{5}{7}} \times \sqrt{\frac{5}{7}} = \sqrt{\frac{25}{49}} = \frac{5}{7}$$

Whence, $\frac{4}{9} \times \frac{5}{7} = \frac{20}{63}$ the answer required.

4. Involve $1 - \sqrt{2}$ to the square, and $1 + \sqrt{3}$ to the cube.

$ \begin{array}{r} 1 - \sqrt{2} \\ 1 - \sqrt{2} \\ \hline 1 - \sqrt{2} \\ -\sqrt{2} + 2 \\ \hline 3 - 2\sqrt{2} \text{ the ans.} \\ \hline \end{array} $	$ \begin{array}{r} 1 + \sqrt{3} \\ 1 + \sqrt{3} \\ \hline 1 + \sqrt{3} \\ \sqrt{3} + 3 \\ \hline 1 + 2\sqrt{3} + 3 \\ 1 + \sqrt{3} \\ \hline 1 + 2\sqrt{3} + 3 \\ \sqrt{3} + 6 + 3\sqrt{3} \\ \hline 10 + 6\sqrt{3} \text{ the Answer.} \\ \hline \end{array} $
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5. Required the square of $2\sqrt[3]{2}$. Ans. $4\sqrt[3]{4}$.

6. What is the cube of $\sqrt{2}$, and the 4th power of $\frac{1}{6}\sqrt{6}$. Ans. $2\sqrt{2}$ and $\frac{1}{36}$

7. Involve $2a - x\sqrt{y}$ to the cube.

$$\text{Ans. } 8a^3 - 12a^2x\sqrt{y} + 4ax^2y - x^3y\sqrt{y}.$$

II. *Evolution of Surds.*

RULE.

Multiply the index of the surd into the index of the root to be extracted, annex the root of the rational part to the result, and it will give the root required.

EXAMPLES.

1. Extract the square root of $16a^{\frac{2}{3}}$

$$\sqrt{16} = 4 \text{ the root of the rational part.}$$

$$a^{\frac{1}{3} + \frac{1}{2}} = a^{\frac{5}{6}} \text{ the root of the surd part.}$$

Wherefore $4a^{\frac{5}{6}}$ is the root required.

2. Extract the Cube root of $\frac{8}{27} \cdot \sqrt[3]{\frac{3}{4}}$

$$\sqrt[3]{\frac{8}{27}} = \frac{2}{3} \text{ and } \sqrt[3]{\frac{3}{4}}^{\frac{1}{3} \times \frac{1}{3}} = \sqrt[3]{\frac{3}{4}}^{\frac{1}{9}}$$

Whence $\frac{2}{3} \sqrt[3]{\frac{3}{4}}$ is the answer.

* To find the square root of a binomial or residual Surd, $\sqrt{A+B}$ or $\sqrt{A-B}$. Take $\sqrt{A^2-B^2}=D$, then will $\sqrt{A+B} = \sqrt{\frac{A+D}{2}} + \sqrt{\frac{A-D}{2}}$. And $\sqrt{A-B} = \sqrt{\frac{A+D}{2}} - \sqrt{\frac{A-D}{2}}$.

For extracting roots higher than the square, no general rule has been discovered. Rules suited to particular cases may be found in Newton's *Arithmetica Universalis*, s' Gravesande's *Matheseos Univers. Elem.* Saunderson's and Maclaurin's *Algebra*, in the *Philosophical Transactions*, and in most works on Algebras

3. Extract the 4th root of $\frac{1}{16}a^{\frac{1}{2}}$

$$\sqrt[4]{\frac{1}{16}} = \frac{1}{2} \text{ and } a^{\frac{1}{2} \times \frac{1}{4}} = a^{\frac{1}{8}}$$

Whence $\frac{1}{2}a^{\frac{1}{8}}$ the answer.

4. Extract the square root of $1 - 2\sqrt{2} + 2$

$$1 - 2\sqrt{2} + 2 \quad (1 - \sqrt{2} \text{ the root.})$$

$$\begin{array}{r} 1 \\ 2 - \sqrt{2} \overline{) (-2\sqrt{2} + 2)} \\ \underline{-2\sqrt{2} + 2} \\ 0 \end{array}$$

5. Extract the square root of $9\sqrt[3]{3}$. *Ans.* $3 \cdot 3^{\frac{1}{6}}$.

6. Required the Cube root of $4^{\frac{1}{3}}$. *Ans.* $4^{\frac{1}{9}}$.

7. Extract the the 4th root of $\frac{1}{81}a^3$. *Ans.* $\frac{1}{3}a^{\frac{3}{4}}$.

8. Required the 5th root of $\frac{2}{3}\sqrt{\frac{x}{a}}$.

$$\text{Ans. } \left(\frac{2}{3}\right)^{\frac{1}{5}} \cdot \left(\frac{x}{a}\right)^{\frac{1}{10}}$$

9. Extract the square root of $a^2 - 4a\sqrt{x} + 4x$.

$$\text{Ans. } a - 2\sqrt{x}$$

12. A Binomial, or residual Surd being given to find a quantity, which multiplied into it, will give a rational product.

RULE*.

Reduce both Surds to a common index, and having changed the sign of one of them, involve this compound quantity to a power whose index is less by 1 than the denominator of the common index.

Prefix the common radical sign to each term of the power, and reject the co-efficients, the result will be the multiplier sought.

EXAMPLES.

1. What quantity multiplying $\sqrt[3]{3} + \sqrt[3]{2}$, will give a rational product, and what will be the product?

$$\begin{array}{r} \text{First. } \sqrt[3]{3} - \sqrt[3]{2} \\ \hline \sqrt[3]{3} - \sqrt[3]{2} \\ \hline \sqrt[3]{9} - \sqrt[3]{6} \\ \quad - \sqrt[3]{6} + \sqrt[3]{4} \\ \hline \sqrt[3]{9} - 2\sqrt[3]{6} + \sqrt[3]{4} \end{array}$$

* Let $\sqrt[n]{a} + \sqrt[n]{b}$ be the given binomial or residual, then will $\sqrt[n]{a^{n-1}} + \sqrt[n]{a^{n-2}b} + \sqrt[n]{a^{n-3}b^2} + \sqrt[n]{a^{n-4}b^3} +$, &c. continued to n terms, be the above rule expressed in general terms.

See Maclaurin's Algebra, p. 113

And rejecting the coefficient 2, the multiplier is $\sqrt[3]{9} - \sqrt[3]{6} + \sqrt[3]{4}$.

$$\begin{array}{r}
 \text{Then } \sqrt[3]{3} + \sqrt[3]{2} \\
 \sqrt[3]{9} - \sqrt[3]{6} + \sqrt[3]{4} \\
 \hline
 \sqrt[3]{27} + \sqrt[3]{18} \\
 - \sqrt[3]{18} - \sqrt[3]{12} \\
 + \sqrt[3]{12} + \sqrt[3]{8} \\
 \hline
 \sqrt[3]{27} \quad * \quad * \quad + \sqrt[3]{8} = 3 + 2 = 5
 \end{array}$$

the rational product.

2. Given $\frac{\sqrt{48}}{\sqrt{7}-3}$ to find what multiplier will make the denominator rational.

First, $(\sqrt{7}-\sqrt{3}) \times (\sqrt{7}+\sqrt{3}) = \sqrt{49}-\sqrt{9} = 7-3=4$ the denominator.

And $\sqrt{48} \times (\sqrt{7}+\sqrt{3}) = \sqrt{336} + \sqrt{144} = 4\sqrt{21} + 12$.

Therefore the multiplier is $\sqrt{7}+\sqrt{3}$.

And the required fraction $\frac{4\sqrt{21}+12}{4} = \sqrt{21}+3$.

3. Find the multiplier for $\sqrt{a}-\sqrt{x}$.

Ans. $\sqrt{a}+\sqrt{x}$.

4. Find the multiplier for $\sqrt[3]{a}+\sqrt[3]{b}$.

Ans. $\sqrt[3]{a^2}-\sqrt[3]{ab}+\sqrt[3]{b^2}$.

5. What quantity multiplying $\sqrt[3]{x^3}+\sqrt[3]{y^3}$, will give a rational product, and what is the product?

Ans. $\sqrt[3]{x^3}-\sqrt[3]{x^2y^3}+\sqrt[3]{x^3y^6}-\sqrt[3]{y^9}$ the multiplier.

And x^3-y^3 the product.

Promiscuous Examples for Practice.

1. Required the sum and difference of $\sqrt{448}$ and $\sqrt{112}$,

Ans. Sum $12\sqrt{7}$.

Diff. $4\sqrt{7}$.

2. Find the product and quotient of $7\sqrt[3]{1000}$ and $5\sqrt[3]{4}$.

Ans. Product $350\sqrt[3]{4}$. Quotient $\frac{7\sqrt[3]{2}}{5}$.

3. Find the sum, difference, product and quotient of $\frac{1}{3}\sqrt[3]{\frac{a}{b}}$ and $\frac{3}{7}\sqrt[3]{\frac{5}{7}}$

Ans. Sum $\frac{1}{3}\sqrt[3]{\frac{a}{b}} + \frac{3}{7}\sqrt[3]{\frac{5}{7}}$. Diff. $\frac{1}{3}\sqrt[3]{\frac{a}{b}} - \frac{3}{7}\sqrt[3]{\frac{5}{7}}$.
 Pro. $\frac{1}{7}\sqrt[3]{\frac{5a}{7b}}$ or $\frac{1}{49b}\sqrt[3]{1715ab^3}$. Quot. $\frac{7\sqrt[3]{7a}}{9\sqrt[3]{5b}}$ or $\frac{7}{45b}\sqrt[3]{875ab^3}$

4. Required the square and square-root of $\frac{1}{9}\sqrt{a^3}$.

Ans. Sq. $\frac{1}{81}a^3$. Square root $\frac{1}{3}\sqrt[3]{a^3}$ or $\frac{1}{3}a^{\frac{1}{3}}$

5. Find the product, quotient, square, and square root of $x^{\frac{1}{m}}$ and $x^{\frac{1}{n}}$.

Ans. Prod. $x^{\frac{m+n}{mn}}$. Quot. $x^{\frac{n-m}{mn}}$. Square $x^{\frac{2}{m}}$
 and $x^{\frac{2}{n}}$. Square root $x^{\frac{1}{2m}}$ and $x^{\frac{1}{2n}}$.

In a given fraction to remove any quantity from the denominator into the numerator, or from the numerator into the denominator.

RULE*.

Any quantity may be removed from one term of a fraction to the other, provided you change the sign of its index.

EXAMPLES.

1. In the fraction $\frac{1}{xy}$ the quantity y is required to be put into the numerator.

$$\text{Ans. } \frac{y^{-1}}{x}$$

2. Change a^{-2} , b^{-3} , $x^{-4}y^{-5}$, and 7^{-2} , into fractions.

$$\text{Ans. } \frac{1}{a^2}, \frac{1}{b^3}, \frac{1}{x^4y^5}, \text{ and } \frac{1}{49}.$$

3. Required the value of $a \times (x-y)^{-\frac{1}{2}}$ and $xy \times (x+y)^{-2}$ in fractions, having affirmative exponents.

$$\text{Ans. } \frac{a}{(x-y)^{\frac{1}{2}}} \text{ and } \frac{xy}{x^2+2xy+y^2}$$

* If the power of any quantity be divided by that quantity, its index will be decreased by 1, as is evident from the nature of division, therefore, $a^3 \div a = (a^{3-1}) a^2$; $a^2 \div a = (a^{2-1}) a^1$; $a^1 \div a = (a^{1-1}) a^0$; $a^0 \div a = a^{-1}$; $a^{-1} \div a = a^{-2}$ and so on. Hence it follows that $(a \div a) = a^0 = 1$; $(a^0 \div a) = a^{-1} = \frac{1}{a}$; $(a^{-1} \div a) = a^{-2} = \frac{1}{a^2}$; $(a^{-2} \div a) = a^{-3} = \frac{1}{a^3}$ &c. from whence the rule will be manifest.

4. Find the values of $\frac{1}{x-1} - \frac{a}{x^2}$ and $\frac{x}{(x-y)^3}$ in integral quantities.

Ans. x , $a x^{-2}$, and $x \times (x-y)^{-3}$.

INFINITE SERIES.

* A series is a rank of quantities or terms usually formed by dividing a simple quantity by a compound one; or by extracting the root of a compound surd, and is such, as being continued would run on infinitely in the manner of a recurring decimal.

As the terms of a series frequently proceed according to some certain law, it will be necessary to obtain only a few of the leading terms from which that law will appear, and by help of which the series may be continued to any length, without actually performing the operation.

* Series are distinguished by particular names according to the nature of their terms, thus,

A series whose terms continually decrease, is called a *converging*, or *approximating* series.

A series whose terms continually increase, is called a *diverging* series.

A series whose terms neither increase nor decrease, is called a *neutral* series.

Thus, $1 - \frac{1}{3} + \frac{1}{6} - \frac{1}{9} + \frac{1}{12}$, &c. is a *converging* series.

$1 - 2 + 3 - 4 + 5$, &c. a *diverging* series, and

$1 - 1 + 1 - 1 + 1$, &c. a *neutral* series. The two latter are of little or no use.

THE BINOMIAL THEOREM*,

For reducing fractions and compound surds to infinite series, and by which all quantities whatever may be multiplied, divided, involved, or evolved.

Let $\overline{P+Q}^m_n$ represent any binomial, where P is the first term, Q the second term divided by the first, $\frac{m}{n}$ the index of the power or root to be found; and let A, B, C, D, &c. each represent the term which immediately precedes that in which it stands.

$$\text{Then will } \overline{P+Q}^m_n = P^m + \frac{m}{n} A Q + \frac{m-m}{2n} B Q^2 + \frac{m-m}{3n} C Q^3 + \frac{m-m}{4n} D Q^4, \text{ \&c.}$$

* This Theorem (of which the rule in page 101 is a particular branch) was discovered by Sir Isaac Newton in 1669, and transmitted in the above form to Mr. Oldenburgh, Secretary to the Royal Society, in a letter dated June 13, 1676, to be by him communicated to M. Leibnetz; although the invention undoubtedly belongs to Newton, yet certain parts of it were known to others before his time. Stifelius, about the beginning of the 16th century, constructed a table of figurate numbers, which shewed, by inspection, the coefficients of the terms of any power of a binomial contained within the limits of the table; and Vieta seems to have well understood the law of the coefficients; but the method of generating them successively one from another was first taught by Mr. Henry Briggs, Savilian professor of Geometry at Oxford, about the year 1600; thus, the theorem, as far as it relates to powers, was complete, wanting only the algebraic form; but whatever others might have discovered, Newton was the first who thought of extracting roots by infinite series; a discovery which has opened a new and extensive field in modern analysis. See Hutton's *Mathematical Tables*, p. 75.

For a demonstration of the binomial Theorem, see Simpson's *Algebra*, p. 227. Ronayne's *Algebra*, p. 215, Landen's *Residual Analysis*, p. 6 and 7. Glenie's *Universal Comparison*; and the observations of Baron Maseres, given in Dr. Hutton's *Mathematical Dictionary*, Vol. 2. p. 732, &c. &c.

If the terms of any binomial quantity, with their proper signs, be substituted in this general form, it will give the answer required.

EXAMPLES.

1. Extract the square root of $a^2 - x^2$ in an infinite series.

$$\text{Here } P = a^2. \quad Q = -\frac{x^2}{a^2}, \quad \frac{m}{n} = \frac{1}{2}, \quad \text{wherefore } P + PQ^{\frac{1}{2}} \\ = a^2 - x^2)^{\frac{1}{2}} \text{ and } P^{\frac{m}{n}} = a^2)^{\frac{1}{2}} = a \text{ the first term} = A.$$

$$\frac{m}{n}AQ = \frac{1}{2} \times A \times -\frac{x^2}{a^2} = -\frac{1}{2} \times a \times \frac{x^2}{a^2} = -\frac{ax^2}{2a^2} = -\frac{x^2}{2a} = B \text{ the second term.}$$

$$\frac{m-n}{2n}BQ = \frac{1-2}{4} \times B \times -\frac{x^2}{a^2} = -\frac{1}{4} \times -\frac{x^2}{2a} \times -\frac{x^2}{a^2} \\ = -\frac{x^4}{8a^3} = C \text{ the third term.}$$

$$\frac{m-2n}{3n}CQ = \frac{1-4}{6} \times C \times -\frac{x^2}{a^2} = -\frac{1}{2} \times -\frac{x^4}{8a^3} \times -\frac{x^2}{a^2} \\ = -\frac{x^6}{16a^5} = D \text{ the fourth term.}$$

$$\frac{m-3n}{4n}DQ = \frac{1-6}{8} \times D \times -\frac{x^2}{a^2} = -\frac{5}{8} \times -\frac{x^6}{16a^5} \times -\frac{x^2}{a^2} \\ = -\frac{5x^8}{128a^7} = E \text{ the fifth term.}$$

$$\frac{m-4n}{5n} EQ = \frac{1-8}{10} \times E \times -\frac{x^2}{a^2} = -\frac{7}{10} \times -\frac{5x^8}{128a^7} \\ \times -\frac{x^2}{a^2} = -\frac{35x^{10}}{1280a^9} = \frac{7x^{10}}{256a^9} = F \text{ the sixth term, \&c.}$$

Wherefore $a^2 - x^2 \Big|_2 = a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} - \frac{7x^{10}}{256a^9}, \&c. \text{ the series required.}$

2. Required the value of $\frac{a}{b+c}$ in an infinite series.

Here $\frac{a}{b+c} = a \times \overline{b+c}^{-1}$, whence $P=b$, $Q=\frac{c}{b}$,

$\frac{m}{n} = -\frac{1}{1}$, and $P^{\frac{m}{n}} = b^{-1} = \frac{1}{b} = A \text{ the first term.}$

$\frac{m}{n} AQ = -\frac{1}{1} \times \frac{1}{b} \times \frac{c}{b} = -\frac{c}{b^2} = B \text{ the second term.}$

$\frac{m-n}{2n} BQ = \frac{-1-1}{2} \times -\frac{c}{b^2} \times \frac{c}{b} = \frac{c^2}{b^3} = C \text{ the third term.}$

$\frac{m-2n}{3n} CQ = \frac{-1-2}{3} \times \frac{c^2}{b^3} \times \frac{c}{b} = -\frac{c^3}{b^4} = D \text{ the fourth term.}$

$\frac{m-3n}{4n} DQ = \frac{-1-3}{4} \times -\frac{c^3}{b^4} \times \frac{c}{b} = \frac{c^4}{b^5} = E \text{ the fifth term.}$

Wherefore the series required is $a \times \left(\frac{1}{b} - \frac{c}{b^2} + \frac{c^2}{b^3} - \frac{c^3}{b^4} + \frac{c^4}{b^5} - \&c. \right)$ or multiplying the series by b , and dividing its coefficient a by the same, the series will be more simply expressed, thus, $\frac{a}{b} \times \left(1 - \frac{c}{b} + \frac{c^2}{b^2} - \frac{c^3}{b^3} + \frac{c^4}{b^4} - \&c. \right)$

3. Find the value of $\frac{1}{(x+y)^2}$ in an infinite series.

Here $\frac{1}{(x+y)^2} = (x+y)^{-2}$, $P=x$, $Q=\frac{y}{x}$, $\frac{m}{n} = \frac{-2}{1}$, whence $P^n = x^{-2} = \frac{1}{x^2} = A$ the first term.

$\frac{m-n}{n} A Q = \frac{-2}{1} \times \frac{1}{x^2} \times \frac{y}{x} = -\frac{2y}{x^3} = B$ the second term.

$\frac{m-n}{2n} B Q = \frac{-3}{2} \times -\frac{2y}{x^3} \times \frac{y}{x} = \frac{3y^2}{x^4} = C$ the third term.

$\frac{m-2n}{3n} C Q = -\frac{4}{3} \times \frac{3y^2}{x^4} \times \frac{y}{x} = -\frac{4y^3}{x^5} = D$ the fourth term.

Wherefore $\frac{1}{(x+y)^2} = \frac{1}{x^2} - \frac{2y}{x^3} + \frac{3y^2}{x^4} - \frac{4y^3}{x^5} + \&c.$ the Answer.

4. Extract the square root of a^2+b in an infinite series.

Ans. $a + \frac{b}{2a} - \frac{b^2}{8a^3} + \frac{b^3}{16a^5} - \frac{5b^4}{128a^7}.$

5. To determine the value of $\frac{ax}{\sqrt{a^2 - x^2}}$ in an infinite series.

$$\text{Ans. } x + \frac{x^3}{2a^2} + \frac{3x^5}{8a^4} + \frac{5x^7}{16a^6} +, \&c.$$

6. Find the value of $\frac{r^2}{r+x}$ in an infinite series.

$$\text{Ans. } r - x + \frac{x^2}{r} - \frac{x^3}{r^2} + \frac{x^4}{r^3} -, \&c.$$

7. Required the value of $\frac{y^2}{(y+x)^2}$ in an infinite series.

$$\text{Ans. } 1 - \frac{2x}{y} + \frac{3x^2}{y^2} - \frac{4x^3}{y^3} + \frac{5x^4}{y^4} -, \&c.$$

8. To find the value of $\frac{1}{y+x}$ in an infinite series.

$$\text{Ans. } \frac{1}{y} - \frac{x}{y^2} + \frac{x^2}{y^3} - \frac{x^3}{y^4} +, \&c.$$

9. Involve 18 or its equal 10+8 to the fourth power by the binomial theorem. Ans. 104976.

10. Find the value of $(x^2 - x^2)^{\frac{1}{2}}$ in an infinite series.

$$\text{Ans. } x^{\frac{2}{3}} \times (1 - \frac{x^2}{5x^2} - \frac{2x^4}{25x^4} - \frac{6x^6}{125x^6} -, \&c.)$$

11. Extract the square root of 2, or, which is the same thing, find the value of $1 + 1^{\frac{1}{2}}$ in an infinite series.

$$\text{Ans. } 1 + \frac{1}{2} - \frac{1}{8} + \frac{1}{16} - \frac{5}{128} + \frac{7}{256} - \&c.$$

12. To find the value of $\sqrt[3]{1-x^3}$ in an infinite series.

$$\text{Ans. } 1 - \frac{x^3}{3} - \frac{x^6}{9} - \frac{5x^9}{81} - \frac{10x^{12}}{243} - \frac{22x^{15}}{729}, \&c.$$

13. Required the value of $\left(\frac{a^2}{(a^2+x^2)^2}\right)^{\frac{1}{2}}$ in an infinite series.

$$\text{Ans. } \frac{1}{a^{\frac{1}{2}}} \times \left(\frac{1}{a^{\frac{1}{2}}} - \frac{2x^2}{3a^{\frac{3}{2}}} + \frac{5x^4}{9a^{\frac{5}{2}}} - \frac{40x^6}{81a^{\frac{7}{2}}} +, \&c. \right)$$

Any series being given to find the several orders of differences.

RULE*.

Subtract the first term from the second, the second from the third, the third from the fourth, and so on; these remainder will form a new series called *the first order of differences*.

In this new series take the first term from the second, the second from the third, the third from the fourth, and so on; the remainders will form another new series called *the second order of differences*.

In like manner find the *third, fourth, fifth, &c.* orders of differences, and proceed till they terminate, or are carried as far as is necessary.

* Let n denote any order of differences d = its first term; and $a, b, c, d, e, \&c.$ the given series; then will $\frac{1}{2}a + nb + \frac{n-1}{2}c + \frac{n-1}{2}d + \frac{n-2}{3}e + \frac{n-2}{3}f + \frac{n-3}{4}g +, \&c. \text{ (to } n+1 \text{ terms)} = d$, according as n is an even or odd number.

EXAMPLES.

1. Required the several orders of differences in the Series 1, 2, 4, 8, 16, 32, 64, &c.

1, 4, 8, 16, 32, 64, &c. *series.*

3, 4, 8, 16, 32, &c. *1st differences.*

1, 4, 8, 16, &c. *2d differences.*

3, 4, 8, &c. *3d differences.*

1, 4, &c. *4th differences.*

&c.

2. Find the several orders of differences in the Series 1, 4, 9, 16, 25, 36, &c.

1, 4, 9, 16, 25, 36, &c. *series.*

3, 5, 7, 9, 11, &c. *first diff.*

2, 2, 2, 2, &c. *second diff.*

0, 0, 0, &c. *third diff.*

3. Find the several orders of differences of 1, 8, 27, 64, 125, 216, &c.

Ans. $\begin{cases} \text{1st } 7, 19, 37, 61, 91, \&c. \\ \text{2d } 12, 18, 24, 30, \&c. \\ \text{3d } 6, 6, 6, 6, \&c. \end{cases}$

4. Find the several orders of differences in the Series 1, 3, 6, 10, 15, 21, &c.

Ans. 1st. 2, 3, 4, 5, &c. 2d. 1, 1, 1, 1, &c.

5. Required the several orders of differences in the Series 1, 4, 8, 13, 19, 26, 34, &c.

Ans. 1st. 3, 4, 5, 6, &c. 2d. 1, 1, 1, &c. 3d. 0, 0, 0.

* To find any term in the series $a, b, c, d, e, \&c.$

RULE.

Let $d^1, d^{11}, d^{111}, d^{1v}, \&c.$ be the first terms of the several orders of differences as found in the last rule, and let the term required to be found be the n^{th} , then will

$$a + \frac{n-1}{1} d^1 + \frac{n-1}{1} \cdot \frac{n-2}{2} d^{11} + \frac{n-1}{1} \cdot \frac{n-2}{2} \cdot \frac{n-3}{3} d^{111} \\ + \frac{n-1}{1} \cdot \frac{n-2}{2} \cdot \frac{n-3}{3} \cdot \frac{n-4}{4} d^{1v} + \&c. = n^{\text{th}} \text{ term required.}$$

EXAMPLES.

1. Required the 10th term of the series 1, 4, 8, 13, 19, 26, &c.

$$\begin{array}{cccccccc} 1, & 4, & 8, & 13, & 19, & 26, & \&c. & \text{Series.} \\ 3, & 4, & 5, & 6, & 7, & & & \&c. \text{ 1st diff.} \\ 1, & 1, & 1, & 1, & & & & \&c. \text{ 2d diff.} \end{array}$$

Here $3 = d^1$ $1 = d^{11}$ $n = 10$ and $a = 1$

$$a + \frac{n-1}{1} d^1 + \frac{n-1}{1} \cdot \frac{n-2}{2} d^{11} = 1 + 9 \times 3 + 9 \times 4 \times 1 \\ = 1 + 27 + 36 = 64 \text{ the 10th term required.}$$

* The Differential method to which this and the following rule belong, is of very general use, especially in the construction of mathematical tables; summation and interpolation of series; finding the areas of curves, &c. It was first made use of by Briggs, in his construction of Logarithms, and has since been treated of by Cotes, Newton, Jones, Stirling, Le Sieur, Jacquier, Simpson, Emerson, and others.

For a demonstration of the above rule, see Emerson's *Differential Method*, prefixed to his *Conic Sections*, p. 12.

2. Find the 20th term of the series 1, 3, 6, 10, 15, 21, &c.

1,	3,	6,	10,	15,	21,	&c.	Series.
	2,	3,	4,	5,	6,	&c.	1st diff.
		1,	1,	1,	1,	&c.	2d diff.

Here $2 = d^1$ $1 = d^{11}$ $n = 20$ $a = 1$, wherefore

$$a + \frac{n-1}{1}d^1 + \frac{n-1}{1} \cdot \frac{n-2}{2}d^{11} = 1 + 19 \times 2 + 19 \times 9 \times 1 \\ = 210 = \text{the 20th term required.}$$

3. Find the 14th term of the series 1, 3, 6, 10, 15, 21, &c. Ans. 105.

4. Find the 12th term of the series 2, 6, 12, 20, 30, &c. Ans. 156.

5. Required the 20th term of the series 1, 8, 27, 64, 125, &c. Ans. 8000.

To find the sum of n terms of the series $a, b, c, d, e, \&c.$

RULE.*

Let $d^1, d^{11}, d^{111}, \&c.$ be the first of the several orders of differences—then will

$$na + n \cdot \frac{n-1}{2}d^1 + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3}d^{11} + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4}d^{111} + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \cdot \frac{n-4}{5}d^{1111} + \&c. = \text{the sum required.}$$

* For a demonstration of the rule, see Emerson's Differential Method, p. 13.

EXAMPLES.

r. To find the sum of n terms of the series 1, 2, 3, 4, 5, &c.

$$\begin{array}{rcll} 1, & 2, & 3, & 4, & 5, & \&c. & \text{Series.} \\ & 1, & 1, & 1, & 1, & \&c. & \text{1st diff.} \\ & 0, & 0, & 0, & 0, & \&c. & \text{2nd diff.} \end{array}$$

Here 1 and 0 are the first terms of the differences.

Whence $a=1$ $d^1=1$ $d^2=0$.

And $na + n \cdot \frac{n-1}{2} d^1 = n + \frac{nn-n}{2} = \frac{nn+n}{2}$ = the sum required.

Thus if the sum of 10 terms be required, then $n=10$ and $\frac{nn+n}{2} = \frac{110}{2} = 55$.

If the sum of 100 terms be required, then $n=100$ and $\frac{nn+n}{2} = 5050$.

To shew the application of this rule.

Suppose 100 stones placed in a straight line a yard from each other, how far will that person travel who brings them one by one to a basket, which stands a yard from the first stone?

Here $a=2$, $d^1=2$, $d^2=0$, $n=100$, and $na + n \cdot \frac{n-1}{2} d^1 = 200 + 100 \times \frac{99}{2} \times 2 = 10100$ yards, or 5 miles 1300 yards.

2. Required the sum of n terms of the series of squares 1, 4, 9, 16, 25, &c.

1, 4, 9, 16, &c. the series.

3, 5, 7, &c. 1st diff.

2, 2, &c. 2d diff.

0, &c. 3d diff.

Here $a=1$ $d^1=3$ $d^{11}=2$ $d^{111}=0$.

And $na+n \cdot \frac{n-1}{2} d^1 + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} d^{11} = n + 3n$.

$$\frac{n-1}{2} + 2n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} = \frac{3n^2-n}{2} + \frac{n^3-3n^2+2n}{3} =$$

$$\frac{2n^3+3n^2+n}{6} = \frac{n \cdot (n+1) \cdot (2n+1)}{6} = \text{the sum required.}$$

Thus suppose the sum of 1000 terms of the above series of squares be required.

Then $n=1000$ and $\frac{n \cdot (n+1) \cdot (2n+1)}{6} =$

$$\frac{1000 \times 1001 \times 2001}{6} = 333833500 \text{ the sum required.}$$

3. To find the sum of n terms of the series of cubes 1, 8, 27, 64, 125, &c.

1, 8, 27, 64, 125, &c. series.

7, 19, 37, 61, &c. 1st diff.

12, 18, 24, &c. 2d diff.

6, 6, &c. 3d diff.

0, &c. 4th diff.

Here $a=1$, $d^1=7$, $d^{11}=12$, $d^{111}=6$, $d^{1111}=0$, and

$$na+n \cdot \frac{n-1}{2} d^1 + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} d^{11} + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} d^{111} = n + 7n \cdot \frac{n-1}{2} + 12n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} +$$

$$6n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} = \frac{7n^3-5n}{2} + 2n^3-6n^2+4n + \frac{n^4-6n^3+11n^2-6n}{4} = \frac{n^4+2n^3+n^2}{4} = \text{the sum required.}$$

Let the sum of 20 terms of the above series of cubes be required to be found:

$$\text{Here } n=20 \text{ and } \frac{n^4+2n^3+n^2}{4} = \frac{160000+16000+400}{4} = \frac{176400}{4} = 44100 \text{ the answer.}$$

4. To find the sum of 100 terms of the series 1, 3, 5, 7, 9, &c. *Ans.* 10000.

5. To find the sum of 1000 terms of the series 1, 2, 3, 4, 5, &c. *Ans.* 500500.

6. How many strokes does the hammer of a common clock strike in 12 hours? *Ans.* 78.

7. Find the sum of n terms of the series 2, 6, 12, 20, 30, &c. *Ans.* $\frac{n \cdot (n+1) \cdot (n+2)}{3}$

8. Required the sum of n terms of the series of Biquadrates, 1, 16, 81, 256, &c. *Ans.* $\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$.

ARITHMETICAL PROPORTION.

Four quantities are said to be in Arithmetical Proportion, when the difference between the first and second, is equal to the difference between the third and fourth.

210 ARITHMETICAL PROPORTION.

Thus, 2, 4, 3, 6, and $a, a+r, b, b+r$, are arithmetically proportional.

An arithmetical progression is a series of quantities which either increase or decrease by the same common difference.

Thus 1, 3, 5, 7, 9, &c. and $a+r, a+2r, a+3r$, &c. are series in arithmetical progression, whose common differences are 2 and r .

* The most useful part of arithmetical proportion is contained in the following Theorems.

Theorem 1. If four quantities are in arithmetical proportion, the sum of the two extremes is equal to the sum of the two means.

* A Synopsis of all the Theorems in Arithmetical Progression.

Let a = the least term, x = the greatest, d = common difference, n = number of terms, s = sum of the series. Then will

$$a = n - (n-1) \cdot d = \frac{2s}{n} - a = \frac{s}{n} - \frac{n-1}{2} d = \sqrt{\left(\frac{1}{2}d + x\right)^2 - 2ds} + \frac{1}{2}d.$$

$$x = a + (n-1) \cdot d = \frac{2s}{n} - a = \frac{s}{n} + \frac{n-1}{2} d = \sqrt{\left(\frac{1}{2}d - a\right)^2 + 2ds} - \frac{1}{2}d.$$

$$d = \frac{x-a}{n-1} = \frac{s-na}{n-1} \cdot \frac{2}{n} = \frac{nx-s}{n-1} \cdot \frac{2}{n} = \frac{x+a \cdot x-a}{2s-a-x}.$$

$$n = \frac{x-a}{d} + 1 = \frac{2s}{a+x} = \frac{\frac{1}{2}d - a + \sqrt{\left(\frac{1}{2}d - a\right)^2 + 2ds}}{d} =$$

$$\frac{\frac{1}{2}d + x - \sqrt{\left(\frac{1}{2}d - a\right)^2 + 2ds}}{d}.$$

$$s = \frac{a+x}{2} n = \frac{a+x}{2} \cdot \frac{x-a+d}{d} = \frac{2a+(n-1) \cdot d}{2} n =$$

$$\frac{2x - (n-1) \cdot d}{2} n.$$

ARITHMETICAL PROGRESSION. 211

Thus if 3, 5, 7, 9, be in arith. proportion.

Then will $3+9=5+7$, being each = 12.

Theorem 2. In any continued arithmetical progression, the sum of the two extremes, and that of any two terms equally distant from them, are equal to each other.

Thus in 2, 4, 6, 8, 10, 12, 14, &c. $14+2$ (the sum of the extremes) $=12+4=10+6=8+8$, being each = 16.

Theorem 3. The last term of an increasing arithmetical series is equal to the common difference multiplied into the number of terms less one, with the first term added to the product.

Thus the 12th term 2, 4, 6, 8, 10, &c. is = $2 \times 11 + 2 = 24$.

Theorem 4. The last term of a decreasing arithmetical series is equal to the difference of the first term, and the product which arises by multiplying the common difference into the number of terms less one.

Thus the 12th term of 24, 22, 20, 18, &c. is $24 - 2 \times 11 = 24 - 22 = 2$.

Theorem 5. The sum of a series of quantities in arithmetical progression, is equal to the sum of the two extremes, multiplied into half the number of terms. *Thus the sum of 1, 2, 3, 4, 5, &c. continued to 12 terms is $(1+12) \times 6 = 78$.*

EXAMPLES

IN ARITHMETICAL PROGRESSION.

1 Required the 100th term of the series 2, 4, 6, 8, &c.

212 ARITHMETICAL PROGRESSION.

By Theorem 3. $2 + 2 \times 99 = 200$, the answer.

2. Required the sum of 10 terms of the series 3, 6, 9, 12, &c.

By Theor. 3. $3 + 3 \times 9 = 30 =$ the last term.

And by Theor. 5. $(3 + 30) \times 5 = 33 \times 5 = 165 =$ the sum required.

3. Required the sum of 20 terms of the decreasing arithmetical series 100, 96, 92, 88, &c.

By Theor. 4. $100 - 4 \times 19 = 100 - 76 = 24 =$ the last term.

And by Theor. 5. $(100 + 24) \times 10 = 124 \times 10 = 1240 =$ the sum required.

4. Required the 12th term of the series 1, 3, 5, 7, &c. Anf. 23.

5. Required the 30th term of the series 100, 97, 94, 91, &c. Anf. 3.

6. Find the sum of 20 terms of the series 1, 4, 7, 10, &c. Anf. 590.

7. Find the sum of 50 terms of the series 9...8, 8...8, 6...8, 4...8, 2...8, &c. Anf. 205.

8. A butcher bought 100 sheep, and gave for the first one shilling, for the second two, for the third three, and so on. What did the 100 sheep cost him?

Anf. £252, 10.

To find any number of arithmetical means between two given quantities.

RULE.

Divide the difference of the given quantities by one more than the required number of means, the quotient will be the common difference, which being continually added to the least term, or subtracted from the greatest, will give the means required.

EXAMPLES.

1. Find two arithmetical means between 4 and 31.

First. $\frac{31-4}{2+1} = \frac{27}{3} = 9 = \text{the common difference.}$

Then $4+9=13=1^{\text{st}}$ mean, $13+9=22=2^{\text{d}}$ mean.

Or, $31-9=22$ and $22-9=13$ the means as before.

2. Find 3 arithmetical means between 2 and 100.

$\frac{100-2}{3+1} = \frac{98}{4} = 24,5 = \text{com. difference.}$

$2+24,5=26,5$ $26,5+24,5=51$ and $51+24,5=75,5$. Therefore the means are 26,5, 51 and 75,5.

3. Required an arithmetic mean between 10 and 100. Ans. 45.

4. Find 2 arithmetical means between 1 and 2. Ans. $1\frac{1}{3}$ and $1\frac{2}{3}$.

5. Find 4 means between a and b , a being the greater.

Ans. $b+\frac{a-b}{5}$, $b+\frac{2.a-b}{5}$, $b+\frac{3.a-b}{5}$ and $b+\frac{4.a-b}{5}$.

OF RATIO and PROPORTION.

1. * Ratio is the relation which one magnitude bears to another of the same kind with respect to quantity.

* No two terms have been more frequently confounded than *Ratio* and *Proportion*, which, in fact, mean quite different things, the former being a comparison of quantities, and the latter a comparison of ratios. The mistake, is, perhaps, in a great measure owing to the

2. If the first of four quantities be exactly as great when compared to the second, as the third is when compared to the fourth, the first is said to have to the second the same ratio which the third has to the fourth.

Thus 2 is as great when compared to 4, as 3 is, when compared to 6.

Therefore 2 has the same ratio to 4, that 3 has to 6.

3. A proportion consists of two equal ratio's. That is, if 2 has the same ratio to 4, that 3 has to 6, this equality or identity of ratio's is called Proportion, and the four quantities are named Proportionals.

4. Two dots interposed between two quantities thus $a : b$ denote their ratio, and four dots thus $::$ denote proportion, or the equality of ratio's.

Thus, if a has to b the same ratio that c has to d, this proportion, or equality of ratio's, is thus expressed, $a : b :: c : d$.

5. Of a ratio, the first term is called the antecedent, and the second the consequent. Thus of four proportionals, the first and third terms were antecedents, and the second and fourth consequents.

6. When a greater quantity is compared with a less, the ratio is called a *Ratio of the greater inequality*; when a less is compared with a greater, it is called a *Ratio of the lesser inequality*, and when equal quantities are compared, the ratio is named the *Ratio of equality*.

vague and unsatisfactory manner in which the terms have been defined even by the best writers; so, in an excellent Mathematical work lately published, we read, that "Ratio is the *proportion* which one magnitude bears to another magnitude of the same kind, with respect to quantity: and immediately after, "Proportion is the equality of *Ratios*." We need not be surprised if confused notions ensue from definitions of this sort.

Thus, $4 : 3$ is a ratio of the greater inequality.

$3 : 4$ - - - - - lesser in equality.

$3 : 3$ - - - - - equality.

7. One quantity is said to be *as* another, or *directly as another*, when one of them being increased, the other must increase, and when one decreases the other decreases also.

8. One quantity is said to be *inversely*, or *reciprocally* as another, when it increases as the other decreases, or decreases as the other increases.

9. If the first of four quantities has the same ratio to the second, which the third has to the fourth. These four quantities are said to be in *Direct* proportion: but if the first is to the second as the fourth to the third, they are said to be in *Inverse*, or *Reciprocal* Proportion.

Thus $2 : 4 :: 6 : 12$ are in *Direct* Proportion.

And $2 : 4 :: 12 : 6$ *Inverse* or *reciprocal* Proportion.

10. A series of quantities is said to be in *Geometrical Progression* when the first has the same ratio to the second, as the second to the third, as the third to the fourth, and so on.

Thus, 2, 4, 8, 16, 32, &c. is a series in *Geometrical Progression*.

11. When every two adjacent terms have the same ratio, the proportion or progression is said to be *continued*: but when the consequent of one ratio differs from the antecedent of the next, the proportion or progression is said to be *discontinued*, *discrete*, or *disjunct*.

Thus, $2 : 4 : 8, 16, 32, 64, \&c.$ are in *continued proportion*.

And $2 : 4, 3, 4, 4, 8, \&c.$ are in *discontinued, or discrete, or disjunct proportion*.

Two or more ratio's being given, to find a ratio equal to their sum.

*RULE.

Multiply the antecedents together for a new antecedent, and the consequents for a new consequent, and divide both by their greatest common divisor; the quotients will be the terms of the required ratio.

EXAMPLES.

1. Given the ratios of $3 : 4$ and $5 : 6$ to find a ratio equivalent to their sum.

First, $3 \times 5 = 15$ the new antecedent.

Then, $4 \times 6 = 24$ the new consequent.

Therefore, $15 : 24$ divided by 3 the greatest common divisor, gives $5 : 8$ the ratio required.

2. Find the sum of the ratio's $2 : 3, 5 : 1$ and $3 : 5$.

$2 \times 5 \times 3 = 30 =$ new antecedent.

$3 \times 1 \times 5 = 15 =$ new consequent.

Therefore, $30 : 15$, or (dividing both by 15) $2 : 1$ is the ratio required.

* A Ratio which is compounded of two or more ratio's is called a compound ratio; if compounded of two equal ratio's, it is called the duplicate ratio; if of three equal ratio's, it is called triplicate ratio; if of four, quadruplicate, &c. Thus the duplicate ratio of a to b is a^2 to b^2 ; the triplicate a^3 to b^3 ; the quadruplicate a^4 to b^4 , &c. Hence squares are to each other in the duplicate, and cubes in the triplicate ratio of their sides or roots.

RATIO and PROPORTION.

217

3. Compound the ratios $a : b, c : d, x : y$, and $4 : 5$.

Ans. $4acx : 5bdy$.

4. Given $x : y, x : z, x : 3$, and $3 : x$.

Ans. $x^2 : yz$.

To resolve a given ratio into two others, one of which is given.

R U L E.

Invert the latter given ratio, and then multiply its terms into the terms of the former, the result will be the terms of the ratio required.

E X A M P L E S.

1. Resolve the ratio $5 : 4$ into two others, one of which is $3 : 2$.

First, $3 : 2$ inverted becomes $2 : 3$.

*Therefore, $5 \times 2 = 10$
 $4 \times 3 = 12$ } whence the required ratio
is $10 : 12$ or $5 : 6$.*

This may be proved by compounding the ratios $3 : 2$ and $5 : 6$ by the former rule; thus,

*$3 \times 5 = 15$
 $2 \times 6 = 12$ } that is $15 : 12$ or $5 : 4$ the very
same as the given ratio.*

2. Resolve $12 : 1$ into two ratios one of which is $100 : 1$.

The latter inverted becomes $1 : 100$.

Therefore, $12 \times 1 = 12$ and $100 \times 1 = 100$: consequently $12 : 100$ or $3 : 25$ is the ratio required.

3. Resolve $x : y$ into two ratios, one of which is $a : b$.

Ans. $bx : ay$.

4. Resolve $48 : 1$ into two other ratios, one of which is $1 : 2$.

Ans. $96 : 1$.

Two ratio's being given to find two others respectively equal to them having a common consequent.

RULE.

Multiply the terms of each ratio into the consequent of the other, and the resulting ratio's will have a common consequent.

EXAMPLES.

1. Reduce the ratio's $3 : 4$ and $2 : 5$ to equal ratio's having a common consequent.

The terms of the first $\times 5$ and those of the second $\times 4$ give $15 : 20$ and $8 : 20$ the ratio's required.

2. Reduce $3 : 8$ and $2 : 1$ to equal ratio's having a common consequent.

The terms of the 1st, $\times 1$ and those of the other $\times 8$; there arises $3 : 8$ and $16 : 8$ the ratio's required.

3. Reduce $1 : 4$ and $3 : 5$ to a common consequent.

Ans. $5 : 20$ and $12 : 20$.

4. Reduce $a : b$ and $x : x$ to a common consequent.

Ans. $ax : bx$ and $bx : bx$.

THEOREMS in PROPORTION.

Theorem 1. If four quantities are proportionals, the product of the two means will be equal to that of the two extremes; and if three quantities are proportionals, the product of the extremes is equal to the square of the mean.

Thus, if $3 : 6 :: 4 : 8$ then will $3 \times 8 = 4 \times 6$; and if $3 : 6 :: 6 : 12$, then will $3 \times 12 = 6 \times 6$.

Theorem 2. If four quantities are proportionals, the quotient of the first term divided by the second, will be equal to that of the third divided by the fourth.

GEOMETRICAL PROGRESSION. 219

Thus, if $3 : 6 :: 4 : 8$ then will $\frac{3}{6} = \frac{4}{8}$.

* Theorem 3. If four quantities are proportionals, the product of the means, divided by either of the extremes will give the other extreme.

Thus if $3 : 6 :: 4 : 8$ then will $\frac{6 \times 4}{3} = 8$ and $\frac{6 \times 4}{8} = 3$.

Theorem 4. In any continued geometrical progression, the product of the two extremes and that of any other two terms equally distant from them will be equal to each other.

Thus in the series 2, 4, 8, 16, 32, 64, &c. $2 \times 64 = 4 \times 32 = 8 \times 16$ being each = 128.

Theorem 5. In any continued geometrical series the greatest term is equal to the least, multiplied into that power of the ratio as is denoted by the number of terms less one, and the least term is equal to the greatest divided by that power of the ratio.

Thus in the series 2, 4, 8, 16, 32, 64, . . . $2 \times 2^5 = 2 \times 32 = 64$ the greater term and $\frac{64}{2^5}$ that is $\frac{64}{32} = 2$ the least.

Theorem 6. The sum of any series in geometrical progression is found by multiplying the greatest term into the ratio, subtracting the least term from the product, and dividing the remainder by the ratio less one. Or it is found by involving the ratio to that

* This Theorem is of the most extensive application being the foundation of the Rule of Three in Arithmetic, and consequently of all the rules which depend thereon, as Practice, Interest, Loss and Gain, Barter, Exchange, Fellowship, Alligation, Position, &c.

220 GEOMETRICAL PROGRESSION.

power which is denoted by the number of terms, multiplying this into the least term, subtracting the least term from the product, and dividing the remainder by the ratio less one.

Thus, the sum of 2, 4, 8, 16, 32, 64, is either

$$\frac{64 \times 2 - 2}{2 - 1} \text{ or } \frac{2 \times 2^6 - 2}{2 - 1}, \text{ being each } = 126.$$

Theorem 7. If four quantities are proportionals, their like powers, roots, multiples, and parts, will also be proportionals.

Thus, if $2 : 4 :: 8 : 16$, then will their squares, cubes, biquadrates, &c. as also their square roots, cube roots, &c. their products, when multiplied into any number whatever, their halves, third parts, fourth parts, &c. be respectively proportionals.

Theorem 8. If four quantities a, b, c, d , or 2, 6, 5, 15, be proportionals, then will any of the following forms of those quantities be proportional.

1. Directly, $a : b :: c : d$, or $2 : 6 :: 5 : 15$.
2. Inversely, $b : a :: d : c \dots 6 : 2 :: 15 : 5$.
3. Alternately, $a : c :: b : d \dots 2 : 5 :: 6 : 15$.
4. By composition, $a : a + b :: c : c + d \dots 2 : 8 :: 5 : 20$.
5. By division, $a : b - a :: c : d - c \dots 2 : 4 :: 5 : 10$.
6. Mixed, $b + a : b - a :: d + c : d - c \dots 8 : 4 :: 20 : 10$.
7. Their/equi multiples, $ra : rb :: c : d \dots 3 \times 2 : 3 \times 6 :: 5 : 15$.
8. Their like parts, $\frac{a}{r} : \frac{b}{r} :: c : d \dots \frac{2}{2} : \frac{6}{2} :: 5 : 15$.

GEOMETRICAL PROGRESSION. 221

*Theorem 9. In any continued geometrical series, the first term has to the third, the duplicate ratio of that which it has to the second, to the fourth the triplicate, to the fifth the quadruplicate, and so on.

Thus, in the series, $a, a^2, a^3, a^4, a^5, \&c.$ a has to a^3 the duplicate ratio, to a^4 the triplicate ratio, &c. of a to a^2 .

EXAMPLES IN GEOMETRICAL PROGRESSION.

1. Required the 10th term of the series 1, 2, 4, 8, 16.

By theor. 5. $1 \times 2^9 = 512$ the answer.

* Let a denote the least term, z the greatest, r the ratio, n the number of terms, s the sum of the progression, and L the logarithm of the quantity before which it stands; then the principal theorems in geometrical progression may be given in general terms, as follow :

$$a = \frac{z}{r^{n-1}} = rz^{-1} : (r-1) = \frac{r-1}{r^{n-1}} s = (s-z) \cdot \frac{1}{z^{n-1}}$$

$$z = ar^{n-1} = \frac{a + (r-1) \cdot s}{r} = \frac{r-1}{r^{n-1}} sr = \frac{s-a}{r} = (s-a) \cdot \frac{1}{a^{n-1}}$$

$$r = \frac{s-a}{s-z} = \frac{n-1}{\sqrt{\frac{z}{a}}} = \frac{s}{a} - 1 = \frac{z}{s-z}$$

$$n = \frac{L_z - L_a}{L_r} + 1 = \frac{L_r(r-1) \cdot (s+a) - L_a s}{L_r} = \frac{L_z - L_a}{L_r(s-a) - L_r(s-z)}$$

$$+ 1 = L_r \frac{r^z}{a} \div L_r r = L_r \frac{r^z}{(r-1)s} \div L_r r.$$

$$s = \frac{r^z - a}{r-1} = \frac{r^n - 1}{r-1} a = \frac{r^n - 1}{r-1} \cdot \frac{z}{\sqrt[n]{z} - \sqrt[n]{a}}$$

222 GEOMETRICAL PROGRESSION.

2. Required the 10th term of the decreasing series 1024, 512, 256, 128, &c.

By theor. 5. $\frac{1024 - 1024}{2^9 - 512} = 2$ the answer.

3. The first term of a geometric series is 1, the ratio 2, and the number of terms 11, to find the sum of the series.

By theor. 6. $\frac{2^{11} \times 1 - 1}{2 - 1} = 2047$ the answer.

4. To find the sum of 12 terms of the series 1, 3, 9, 27, &c. *Ans.* 265720.

5. The first term of a geometrical series is 1, the ratio 2, and the number of terms 10, to find the sum of the series. *Ans.* 1023.

6. A man bought a horse, for which he was to give a farthing for the first nail, two for the second, four for the third, &c. in geometrical proportion; now there were 4 shoes, and 8 nails in each, what did the horse cost? *Ans.* £4473924. 5s. 3½d.

7. To find a mean proportional between 2 and 32.

By theor. 1. $\sqrt{2 \times 32} = \sqrt{64} = 8$ for 2 : 8 :: 8 : 32.

8. Find a mean proportional between a and b , and between 1 and 10000. *Ans.* $\sqrt{ab}$ and 100.

To find any number of mean proportionals between two quantities.

RULE.

Divide the greater quantity by the less, and extract that root of the quotient whose index is 1 more than

GEOMETRICAL PROGRESSION. 223

the required number of means; this root will be the common ratio; then multiply the least quantity into the ratio and the product will be the first mean: multiply this product into the ratio for the second mean, and this last into the ratio for the third, and so on till all the required means are produced; or divide the greater quantity continually by the ratio, the quotients will be the means as before.

EXAMPLES.

1. Find 2 mean proportionals between 2 and 128:

First, $\sqrt[3]{\frac{128}{2}} = \sqrt[3]{64} = 4$ the ratio.

Then $2 \times 4 = 8$ the first mean, and $8 \times 4 = 32$ the second mean.

Or thus, $\frac{128}{4} = 32$, and $\frac{32}{4} = 8$ the means as before.

*Therefore, 8 and 32 are the mean proportionals required.
For $2 : 8 : 32 : 128$ are proportionals.*

2. Find 4 mean proportionals between 1 and 243.

First, $\sqrt[5]{\frac{243}{1}} = 3$ the ratio, consequently $1 \times 3 = 3$ the first, $3 \times 3 = 9$ the second, $3 \times 9 = 27$ the third, $3 \times 27 = 81$ the fourth.

Therefore, 3, 9, 27, and 81, are the mean proportionals required.

3. Find 2 mean proportionals between 4 and 256.
Ans. 16 and 64.

4. Required 3 mean proportionals between 2 and 162.
Ans. 6, 18, and 54.

324 HARMONICAL PROPORTION.

5. Required 5 mean proportionals between 1 and 64.
Ans. 2, 4, 8, 16, and 32.

HARMONICAL PROPORTION.

*Three quantities are said to be in harmonical or musical proportion, when the first has the same ratio to the third, as the difference between the first and second has to the difference between the second and third.

Thus, 2, 3, and 6, are in harmonic proportion, for
 $2 : 6 :: 3 - 2 : 6 - 3.$

Also, four terms are in harmonical proportion, when the first is to the fourth, as the difference between the first and second, is to the difference between the third and fourth.

Thus, 3 : 5 : 6, and 18, are harmonically proportional.

Theorem 1. If double the product of two quantities be divided by their sum, the quotient will be a harmonic mean proportional between them.

Thus, the harmonic mean between a and b is $\frac{2ab}{a+b}.$

* The reciprocals of an Arithmetical Progression, are in Harmonical Progression, and conversely. Also, if there be taken an arithmetical mean, and a harmonical mean between any two quantities, these four will be in geometrical proportion:

The following beautiful and simple comparison of the three kinds of proportionals was first given by Pappus. Let a , b , and c be the first, second, and third terms, then in the

Arithmeticals, $a : a$
 Geometricals, $a : b$
 Harmonicals, $a : c$ } $:: a - b : b - c.$

There is also this remarkable difference between the three kinds of proportion, viz. that from any given number there can be raised a continued arithmetical series, increasing *ad infinitum*, but not decreasing; while the harmonical can be decreased *ad infinitum*, but not increased; and the geometrical admits of both.

HARMONICAL PROPORTION. 225

Theorem 2. If the product of two quantities be divided by the difference between double the first and the second, the quotient will be a harmonical third proportional to the two quantities.

Thus, the harmonic third proportional to a and b is

$$\frac{ab}{2a-b}.$$

Theorem 3. If three terms are given a fourth harmonical proportional may be found, by dividing the product of the first and third, by the difference between double the first and the second.

Thus, if a , b , and c are given, then the 4th harmonic proportional is

$$\frac{ac}{2a-b}.$$

EXAMPLES.

1. To find a harmonical mean between 4 and 12.

By theor. 1. $\frac{2 \times 4 \times 12}{4 + 12} = \frac{96}{16} = 6$ the mean required.

For $4 : 12 :: 6 - 4 : 12 - 6.$

2. To find a harmonic third proportional to 4 and 6.

By theor. 2. $\frac{4 \times 6}{2 \times 4 - 6} = \frac{24}{2} = 12$ the answer.

For $4 : 12 :: 6 - 4 : 12 - 6.$

3. To find a fourth term harmonically proportional to 2, 3, and 6.

By theor. 3. $\frac{2 \times 6}{2 \times 2 - 3} = 12$ the answer.

For $2 : 12 :: 3 - 2 : 12 - 6.$

A general view of the nature, formation, and roots of Equations.

A simple equation is that which contains the unknown quantity in the first power only.

A quadratic equation is that which contains the second power or square of the unknown quantity.

A cubic equation is that which contains the third power or cube of the unknown quantity.

A biquadratic equation is that which contains the fourth power of the unknown quantity.

An equation of the fifth power or degree is that which contains the fifth power of the unknown quantity; one of the sixth degree contains the sixth power, and so on.

An equation which contains one power only of the unknown quantity is called *pure*, as $x^2=ab$ a *pure* quadratic; $ax^3=b$ a *pure* cubic, &c. but if the equation contains several powers of the unknown quantity it is called *affected*, *adaffected*, or *compound*, thus $x^2+ax=b$ is an *affected*, *adaffected*, or *compound* quadratic; $x^3-ax^2+bx=c$ an *affected* cubic, $x^4+ax^3+bx^2+cx-d=0$ an *affected* biquadratic, &c.

The root of an equation is the value of the unknown quantity contained in it, and is such that being substituted into the equation instead of the letter denoting that unknown quantity, it will make all the terms vanish, or (which is the same thing) the equation will become equal to nothing.

The roots of equations are either real or imaginary.

Real roots are those whose values can be known and understood, and are either positive or negative.

Imaginary or impossible roots are those to which no real value can be assigned, such are the even roots of negative quantities as $\sqrt{-a^2}$, $\sqrt[4]{-a^4}$, $\sqrt[6]{-5}$, $\sqrt{-4}$, &c.

*All equations of superior degrees are generated by the multiplication of those of inferior degrees.

Thus, a *quadratic* equation is formed by the multiplication of two simple equations, a *cubic* equation by the multiplication of three simple ones, a *biquadratic* equation by four simple ones, and so on.

For example, suppose the unknown quantity x to be equal to a , b , c , d , &c. that is, $x=a$, $x=b$, $x=c$, $x=d$; these, by transposition, become $x-a=0$, $x-b=0$, $x-c=0$, $x-d=0$, which being multiplied together, we have

$$x-a=0$$

$$x-b=0$$

$$\left. \begin{array}{r} x-a \\ -b \end{array} \right\} x+ab=0 \text{ a quadratic.}$$

$$x-c=0$$

$$\left. \begin{array}{r} x^2-a \\ -b \\ -c \end{array} \right\} \left. \begin{array}{r} +ab \\ +ac \\ +bc \end{array} \right\} x-abc=0 \text{ a cubic.}$$

$$x-d=0$$

$$\left. \begin{array}{r} x^3-a \\ -b \\ -c \\ -d \end{array} \right\} \left. \begin{array}{r} +ab \\ +ac \\ +ad \\ +bc \\ +bd \\ +cd \end{array} \right\} \left. \begin{array}{r} -abc \\ -abd \\ -acd \\ -bcd \end{array} \right\} x+abcd=0 \text{ a biquadratic}$$

&c.

* This method of generating affected equations was discovered by Mr. Thomas Harriott, and given in his *Praxis Artis Analyticae*, published by Warner in 1631. See the *Arithmetica Universalis*, Dr. Wallis's *Algebra*, Maclaurin's *Algebra*, and Dr. Hutton's *Mathematical Dictionary*, p. 89.

In the above example, two simple equations (whose roots are a and b) multiplied together produce the quadratic; three simple equations (whose roots are a , b , and c) produce the cubic; four simple equations (whose roots are a , b , c , and d) produce the biquadratic, &c.

Hence it appears that every equation has as many roots, as there are simple equations which produce it, or as many roots as the unknown quantity has dimensions and no more; for if any of the roots a , b , c , d , be substituted for x in the above biquadratic equation, the terms will all vanish, and the whole become equal to nothing, which will not be the case when any other quantity besides these is substituted for the root.

It appears also from the inspection of these equations, that the coefficient of the first term in each is unity.

The coefficient of the second term in each is *the sum of all the roots (a , b , &c.) with contrary signs.*

The coefficient of the third term in each, is *the sum of all the products that can arise by combining the roots two and two.*

The coefficient of the fourth term is *the sum of all the products that can arise by combining the roots three by three, and so on.*

The last term is *the product of all the roots with contrary signs*, and this reasoning will hold whether the roots be positive or negative.

It likewise appears by inspecting the foregoing equations that the signs of all the terms are alternately $+$ and $-$.

The first term is positive and some pure power of x .

The second term is some power of x multiplied into $-a$, $-b$, $-c$, &c. and since these quantities are negative, the term itself must be negative also.

The third term has for its coefficient, the product of any two of these quantities $-a, -b, -c, -d$. and is therefore positive, since $- \times -$ produces $+$.

For the same reason the next coefficient must be negative, being the products of three negative quantities: and the next being the products of four must be positive.

It appears likewise from the equations before referred to, that *when the roots are all positive, the signs of the terms will be plus and minus alternately; and conversely, when the signs are alternately plus and minus, the roots are all positive.*

Let now the roots of the above mentioned equations be considered as negative, that is $x = -a, x = -b, x = -c, x = -d$, these transposed become $x + a = 0, x + b = 0, x + c = 0, x + d = 0$. and their product $(x + a) \cdot (x + b) \cdot (x + c) \cdot (x + d)$ will form an equation in which all the terms will be positive, as is evident, seeing all the factors which produce it are positive.

Hence, *when all the roots of an equation are negative, it is plain there can be no change in the signs of that equation; and conversely, when the signs of all the terms of an equation are alike, the roots are all negative.*

If an equation be generated (like the foregoing) having some of its roots $+$ and others $-$ it will appear that *there will be as many changes in the signs of its terms from $+$ to $-$ or from $-$ to $+$ as the equation has positive roots; and conversely, the equation will have as many positive roots as there are changes in the signs of its terms, and all the rest will be negative.*

It follows from what has been said, that Quadratic Equations have two roots, which may be either both positive, or both negative, or one affirmative and one negative.

Cubic Equations have three roots, which may be *all positive*, or *all negative*; or *two of them may be positive and one negative*, or *two negative and one positive*.

These observations may be extended to Equations of all degrees, and from them rules may be derived for solving those equations.

Having one root of an equation given, to depress the equation to a lower degree.

RULE.

Transpose all the terms to one side of the equation, by which it will become $= 0$. Change the sign of the root and connect it to the unknown quantity. Divide the given equation by this compound quantity, and the quotient will be an equation ($= 0$) depressed to a lower dimension.

EXAMPLES.

1. Given one root $= 3$ of the equation $x^3 - 9x^2 + 26x - 24 = 0$ to depress it to a quadratic.

$$x - 3) x^3 - 9x^2 + 26x - 24 \quad (x^2 - 6x + 8 = 0 \text{ the quad-} \\ x^3 - 3x^2 \quad \text{[ratic required.]}$$

$$\underline{-6x^2 + 26x}$$

$$\underline{-6x^2 + 18x}$$

$$8x - 24$$

$$\underline{8x - 24}$$

2. Given $x = 2$ in the equation $x^2 - 5x + 6 = 0$ to depress it.

$$x - 2) x^2 - 5x + 6 \quad (x - 3 = 0 \text{ the eq. required.}$$

RESOLUTION of EQUATIONS, &c. 235

3. Given $x=5$ in the equation $x^4-14x^3+71x^2-154x+120=0$ to deprefs it to a lower degree.

$$\text{Ans. } x^3-9x^2+26x-24=0$$

4. Given $x=1$ to deprefs the equation $x^4-x^3-19x^2+49x-30=0$.

$$\text{Ans. } x^3-19x+3=0$$

The RESOLUTION of EQUATIONS of SEVERAL DIMENSIONS by APPROXIMATION.

* RULE.

Find by trial two numbers as near the true root as possible.

* The rule is founded on this supposition, that the first error is to the second, as the difference between the true and first supposed number, is to the difference between the true and second supposed number, which though not accurately true in all cases, yet it will be very nearly so.

That the rule is true according to the supposition, may be thus demonstrated.

Let a and b be the two suppositions; A and B their results produced by like operations; it is required to find the number from which N is produced by a similar operation.

Let x = number required $N-A=r$, $N-B=s$, then by supposition $r:s::x-a:x-b$; and by division $r-s:s::b-a:x-b$, that is $\frac{(b-a)s}{r-s}=x-b$, which is the rule when the assumed quantities are both too little.

Next let A and B be both greater than N , then $N-A=-r$, $N-B=-s$, but $-r:-s::+r:+s$, wherefore $r-s:s::a-b:b-x$, or $\frac{(a-b)s}{r-s}=b-x$, which is the rule when both quantities are too great.

Again, let one result A be too little, and the other B too great; then will r be positive, and s negative, and in this case $r+s:(-s \text{ or which is the same}) s::a-b:b-x$.

Substitute these numbers for the unknown quantity in the given equation, and mark the errors which arise from each of them.

Multiply the difference of the two numbers found by trial, into the least error, and divide the product by the difference of the errors when they are alike, but by their sum when they are unlike, and add the quotient to the number belonging to the least error, when that number is too little, but subtract it when too great, the result will be the root, *nearly*.

Take this root and the nearest of the former, (or any nearer number) and proceeding in like manner a root will be obtained *nearer* than before, and by again

Whence $\frac{(a-b).s}{r+s} = b-x$ which is the rule when the errors are unlike.

Various approximating rules have been given by Stevinus, Vieta, Newton, Wallis, Raphson, Halley, De Lagny, Simpson, and others, but there is no method so easy and general as the above, which will resolve the most difficult forms of Equations of all degrees, however embarrassed by surds, compound quantities, &c. without any previous reduction, and for finding the roots of Exponential Equations, it is, perhaps, the best and easiest rule extant.

There have been discovered particular rules for Cubic and Biquadratic Equations as follow.

A Rule for Cubics, commonly called Cardan's Rule, discovered first by Scipio Ferreus, in 1505; and afterwards by Nicholas Tartalea, who communicated it to Cardan, by whom it was published in 1545.

A Rule for Biquadratics invented by Lewis Ferrari, about 1540, and published in Cardan's Algebra.

Another given by Descartes, in his Geometry, 1637.

Another by M. Daniel Bernoulli, and

Another by Mr. Euler, both given in the Petersburg Acts, and likewise

Another by Mr. Thomas Simpson, in his Algebra, p. 150.

But no general rule has yet been discovered for Equations higher than the 4th power.

repeating the operation the root will be had still *neater*, and so on to any degree of exactness.

Having found one root, depress the equation by the preceding rule, and then find another root; depress this last equation, and find a third root, and so on till the equation is brought to a quadratic whose roots may be found by the rule for Quadratic Equations.

EXAMPLES.

1. Given $x^3 + x^2 + x = 20$ to find the roots, or values of x .

Here it will appear that x is some number between 2 and

3. Assume those numbers for x , and the operation will stand thus:

1st Supposition.

2^d Supposition.

2	—	x	—	3
4	—	x^2	—	9
8	—	x^3	—	27
14	.	Sums	.	39
-6	.	Errors	.	+19

The sum of the errors $(19 + 6) = 25$.

Difference of numbers found by trial = 1.

Least error = 6.

Therefore $\frac{1 \times 6}{25} = \frac{6}{25}$, 24 is the correction, and $x = 2, 24$ nearly.

234 RESOLUTION of EQUATIONS

Secondly, let 2,2 and 2,3 be each put for x , and repeat the operation thus :

3d Supp.						4th Supp.
2,2	.	.	.	.	x	2,3
4,84	.	.	.	.	x^2	5,29
10,648	.	.	.	.	x^3	12,167
17,688	.	.	.	Sums	.	19,757
—2,312	.	.	.	Errors	.	—0,243

Their difference ($=2,312-0,243$) $=2,069$.

Difference of numbers, $=,1$. Least error $=0,243$.

Therefore, $\frac{1 \times ,243}{2,069} = ,01174$ is the correction.

And $x = (2,3 + ,01174) = 2,32174$ extremely near.

Now in order to find the other two roots, depress the given equation. We have $x^3 + x^2 + x - 20 = 0$ for a dividend, and $x - 2,31174 = 0$ for a divisor; but because the second and third decimal figures in the divisor are small, and the work, only an approximation, $x - 2,3$ may be used, and the rest omitted, thus :

$x - 2,3$) $x^3 + x^2 + x - 20$ ($x^2 + 3,3x + 8,59 = 0$ a quadratic.

$$\begin{array}{r}
 x^3 - 2,3x^3 \\
 \hline
 3,3x^2 + x \\
 3,3x^2 - 7,59x \\
 \hline
 8,59x - 20 \\
 8,59x - 19,757 \\
 \hline
 - ,243 \\
 \hline
 \end{array}$$

The two roots of this equation ($x^2 + 3,3x + 8,59 = 0$) being found by the rule for quadratics, are $-1,65 \pm \sqrt{-5,8675}$.

Wherefore the three roots of the given equation are $2,31174, -1,65 + \sqrt{-5,8675}$, and $-1,65 - \sqrt{-5,8675}$ nearly, the two last of which are impossible.

2. Given $x^4 + 2x^3 + 3x^2 + 4x = 70$ to find x .

Let 2 and 3 be the supposed numbers, then,

1st Sup.		2d Sup.	
8	—	$4x$	— 12
12	—	$3x^2$	— 27
16	—	$2x^3$	— 54
16	—	x^4	— 81
52	. . . Sums	. . .	174
-18	. . . Errors	. . .	+104

And $\frac{1 \times 18}{18 + 104} = \frac{18}{122} = .147 =$ the correction, wherefore $x = 2,147$ nearly.

Now for a nearer approximation let 2,1 and 2,2 be the supposed numbers, and we shall have,

3d Sup.		4th Sup.	
8,4	. . . $4x$	. . .	8,8
13,23	. . . $3x^2$	. . .	14,52
18,522	. . . $2x^3$	. . .	21,296
19,4481	. . . x^4	. . .	23,4256
59,6001	. . . Sums	. . .	68,0416
-10,3999	. . . Errors	. . .	-1,9584

236 RESOLUTION of EQUATIONS, &c.

Diff. numbers $(=2,2-2,1)=,1.$

Diff. errors $(=10,3999-1,9584)=8,4415.$

Wherefore $\frac{1 \times 1,9584}{8,4415} = ,0232$ *the correction.*

Whence $x=(2,2+,0232=) 2,2232$, *and by depressing the equation, &c. as in the preceding example, the other roots will be obtained.*

3. Given $\sqrt{1+x^2} + \sqrt{2+x^2} + \sqrt{3+x^2} = 10$ to find x .

Let 3 and 4 be the supposed numbers,

<i>1st Sup.</i>		<i>2d Sup.</i>
3,16227	$\sqrt{1+x^2}$	4,12310
3,31662	$\sqrt{2+x^2}$	4,24264
3,46410	$\sqrt{3+x^2}$	4,35889
9,94299	<i>Sums</i>	12,72463
-0,05701	<i>Errors</i>	+2,72463

Wherefore $\frac{1 \times ,05701}{2,78164} = ,0205$ *the correction.*

And $x=3,0205$ *nearly, and by repeating the operation a nearer value of* x *will be obtained.*

4. Given $x^2 + 20x = 100$ to find x .

Ans. $x=4,142135.$

5. Given $x^2 - 2x = 5$ to find x .

Ans. $x=2,094251.$

6. Given $x^3 + x^2 + x - 100 = 0$ to find x .

Ans. $x=4,264429.$

7. Given $2x^3 + 3x^2 + 4x = 100$ to find x .

Ans. $x=3,0896.$

EXTRACTION of ROOTS.

237

8. Given $x^4 - 3x^2 - 75x = 10000$ to find x .

Ans. $x = 10,2615$.

9. Given $x^5 + 2x^4 + 3x^3 + 4x^2 + 5x = 54321$ to find x .

Ans. $x = 8,4144$.

10. Given $\frac{20x}{\sqrt{16+5x+x^2}} + \frac{x\sqrt{5+x^2}}{25} = 34$ to find x .

Ans. $x = 20,17$.

11. Given $\sqrt[3]{1+x^3} + \sqrt[3]{1+x^2} + \sqrt[3]{1+x} = 6,5$ to find x .

Ans. $x = 3,036$.

EXTRACTION of ROOTS.

To extract the root of any pure power in numbers by approximation.

RULE*.

Let N be the given power whose root is required, n the index of that power, x the root required.

* Demonstration of the rule. Let N = given number, n = index of the power, r = nearest rational root, v = remaining part. $x = r + v$ = true root.

Then $N = (r+v)^n = r^n + nr^{n-1}v + n \cdot \frac{n-1}{2} r^{n-2}v^2 + \&c.$

And $\frac{N-r^n}{nr^{n-1}} = v + \frac{n-1}{2} \cdot \frac{v^2}{r} \&c.$ where rejecting $\frac{n-1}{2} \cdot \frac{v^2}{r}$ on

account of its smallness, v may be considered as nearly $= \frac{N-r^n}{nr^{n-1}}.$

Find by trials a number nearly equal to x , and call it r , and let its power (which will be nearly equal to N) be called A , then will $(n+1) A + (n-1) \cdot N : (n+1) N + (n-1) \cdot A :: r : x$ nearly; or which is the same thing,

$$x = \frac{(n+1) \cdot N + (n-1) \cdot A}{(n+1) \cdot A + (n-1) \cdot N} \times r \text{ nearly.}$$

To find a nearer value of x , put $r =$ the value of x last found, $A =$ its power, and repeat the operation as often as may be thought necessary.

EXAMPLES.

1. Let $x^4 = 18$ be given to find the value of x .

Here $N = 18$, $n = 4$. Let $2 = r$, then $(2^4 =) 16 = A$.

But $N - r^n = nr^{n-1}v + n \cdot \frac{n-1}{2} r^{n-2}v^2 + \&c. = \left(nr^{n-1} + n \cdot \frac{n-1}{2} r^{n-2}v \right) \times v$, and by substituting the former value of v in this

equation, we have $N - r^n = \left(nr^{n-1} + \frac{n-1}{2} \cdot \frac{N-r^n}{r} \right) \times v =$

$$\left(\frac{2nr^n + n-1 \cdot N - nr^n}{2r} \right) \times v = \left(\frac{n+1 \cdot r^n + n-1 \cdot N}{2r} \right) \times$$

v . consequently $v = \frac{(N - r^n) \times 2r}{n+1 \cdot r^n + n-1 \cdot N}$. and $r + v (=x) = r +$

$$\frac{(N - r^n) \times 2r}{n+1 \cdot r^n + n-1 \cdot N} = \frac{n+1 \cdot N + n-1 \cdot r^n}{n+1 \cdot r^n + n-1 \cdot N} \times r \text{ which is the rule.}$$

This rule is preferable to the formulæ given by Raphson, Halley, and De Lagny, and will be more easily remembered; it was invented by Dr. Hutton, and published in his Mathematical Tracts.

$$\text{Then } (n+1).A = 5 \times 16 = 80$$

$$(n-1).N = 3 \times 18 = 54$$

$$\text{Their sum } \underline{134}$$

$$\text{Also, } (n+1).N = 5 \times 18 = 90$$

$$(n-1).A = 3 \times 16 = 48$$

$$\text{Their sum } \underline{138}$$

$$\text{Therefore, } 134 : 138 :: 2 : x.$$

$$\text{Whence } x = \frac{2 \times 138}{134} = \frac{276}{134} = 2,06 = \text{the root nearly.}$$

2. Let $x^3 = 300$ be given to find x .

Here $N = 300$, $n = 5$, assume $3 = r$, then $(3^3 =) 243 = A$.

$$(n+1).A + (n-1).N = 6 \times 243 + 4 \times 300 = 1458 + 1200 = 2658 = \text{first term.}$$

$$(n+1).N + (n-1).A = 6 \times 300 + 4 \times 243 = 1800 + 972 = 2772 = \text{second term.}$$

$$\text{Therefore, } 2658 : 2772 :: 3 : x.$$

$$\text{Whence } x = \frac{2772}{2658} \times 3 = 3,12 \text{ nearly.}$$

Now in order to find a nearer value of x , assume $r = 3,12$.

$$\text{Then } (3,12)^3 = 295,64665, \&c. = A, \text{ and}$$

$$(n+1).A + (n-1).N = 6 \times 295,64665 + 4 \times 300 = 1773,87990 + 1200 = 2973,8799 = \text{the first term. And}$$

$$(n+1).N + (n-1).A = 6 \times 300 + 4 \times 295,64665 = 1800 + 1182,5866 = 2982,5866 = \text{the second term.}$$

Therefore $2973,8799 : 2982,5866 :: 3,12 : x$

Or $x = \frac{2982,5866}{2973,8799} \times 3,12 = 3,1291104 = \text{the}$
root required extremely near.

3. Given $nx^4 = 90$ to find the value of x .

Ans. $x = 3,0807$.

4. Given $x^3 = 10$ to find x .

Ans. $x = 2,15443$.

5. Given $x^5 = 2$ to find x .

Ans. $x = 1,148699$.

6. Given $x^6 = 21035,8$ to find x .

Ans. $x = 5,254037$.

7. Extract the 6th, 7th, 8th, and 9th roots of 2.

Ans. 1,122462 . . . 1,104089 . . . 1,090508 and
1,080059 respectively.

PROBLEMS,

With the Methods of resolving them pointed out.

1. What number is that whose half, third, fourth, and sixth parts being added together, the sum exceeds the given number by 1?

Let $x = \text{the number}$, then $\frac{x}{2} + \frac{x}{3} + \frac{x}{4} + \frac{x}{6} = x$
 $+ 1$ which cleared of fractions, &c. gives $x = 4 = \text{the}$
number required.

2. To divide the number 72 into four parts, so that the first may exceed the second by 10, the third by 18, and the fourth by 20.

Let $x = \text{the first part}$, then $x - 10 = \text{second}$, $x - 18 =$
third, $x - 20 = \text{fourth}$, and their sum $4x - 48 = 72$,
whence $x = 30$.

PROBLEMS.

241

3. A person being asked what o'clock it was, answered that the time which had passed from noon was equal to $\frac{3}{5}$ of what remained till midnight, what was the hour?

Let x = the time from noon, then $12 - x$ = time remaining to midnight, whence $x = \frac{3}{5} \times (12 - x) = \frac{36 - 3x}{5}$ (which reduced) $= 4\frac{1}{2}$, or half past four in the afternoon.

4. The paving of a square at 2s. a yard, cost as much as the inclosing it at 5s. a yard, the side of that square is required.

Let x = side, $4x$ = yards of inclosure, x^2 = yards of pavement, $(4x \times 5) 20x$ = price of inclosing, $(x^2 \times 2) 2x^2$ = price of paving, hence $2x^2 = 20x$ and $x = 10$.

5. Out of a cask of wine, which had leaked away one third, 21 gallons were drawn; and then, being gauged, it was found to be half full; how much did it hold?

Let x = No. $\frac{x}{3}$ = leaked, whence $21 + \frac{x}{3}$ = quantity taken away.

Therefore $21 + \frac{x}{3} = \frac{x}{2}$ and $x = 126$.

6. After A had won a shilling of B, he had as many shillings as B, left; but had B won a shilling of A, he would have had twice as many as A would have had left, how many had each?

Let $x = A$'s, $y = B$'s, then $x + 1 = y - 1$; $y + 1 =$
 $(x - 1 \times 2 =) 2x - 2$; $y = x + 2$ (Eq. 1) $= 2x - 3$
 (Eq. 2) $\therefore x = 5, y = 7$.

7. A labourer engaged to serve for 40 days on these conditions, that he should receive 2cd. for every day he worked, and forfeit 8d. for every day he did not work. Now at the end of the time he received £1. 1s. 8d. how many days did he work, and how many was he idle?

Let $x =$ No. of days he worked, $40 - x =$ No. idle,
 $(20 \times x =) 20x =$ sum earned; $(40 - x \times 8 =) 320 -$
 $8x =$ sum forfeited, $20x - 320 - 8x = 380d$ (£1. 1s. 8)
 $x = 25$. $40 - x = 15$.

8. Four persons, A, B, C, and D, divide 1944 guineas, their respective shares were as the numbers 2, 3, 4, and 5, how many did each receive?

Let $2x = A$'s share, then $3x = B$'s, $4x = C$'s, $5x =$
 D 's.

G. s.

Their sum $14x = 1944$, $x = 138$ 18.

9. A draper sold $\frac{1}{4}$ of a piece of cloth at 5s. $\frac{1}{2}$ of it at 4s. and $\frac{1}{6}$ of it at 4s. 6d. per yard, for which he received eight guineas. How many yards did the piece contain?

Let $x =$ No. of yards, $\frac{x}{4} \times 5 + \frac{x}{5} \times 4 + \frac{x}{6} \times \frac{9}{2}$
 $= 168$.

Or $56x = 3360 \therefore x = 60$.

* Three dots $\therefore$ denote the word therefore.

PROBLEMS.

243

10. When first the marriage knot was tied
Betwixt my wife and me,
My age to her's we found agreed,
As nine doth unto three:
But after ten, and half ten years
We man and wife had been,
Her age approach'd as near to mine,
As eight is to sixteen.
Now tell me, if you can, I pray,
Our ages on the marriage day.

Let x = man's age, $9 : 3 :: x : \frac{3x}{9} = \frac{x}{3}$ = woman's,

and $x + 15 : \frac{x}{3} + 15 :: 16 : 8$, or $2 : 1 \therefore x + 15 =$

$$\frac{2x}{3} + 30, x = 45. \frac{x}{3} = 15.$$

11. A hare is 50 leaps before a greyhound and takes 4 leaps to the greyhound's three: but two of the greyhound's leaps are as much as 3 of the hare's; how many leaps must the greyhound take to catch the hare?

Let x = greyhound's leaps, y = hare's in the same time,

$$3 : 4 :: x : y \therefore 3y = 4x, y = \frac{4x}{3}, \text{ but } 50 + \frac{4x}{3} =$$

the hare's leaps in the whole, *conseq.* $2 : 3 :: x : 50$

$$+ \frac{4x}{3} \therefore \left(50 + \frac{4x}{3}\right) \times 2 = 3x, \text{ or } 100 + \frac{8x}{3} = 3x.$$

$$\therefore x = 300, y = 400.$$

12. Bought 3 books whose prices were as 12, 5, and 1 respectively, now if the price of the first be doubled, the second trebled, and the third quadrupled, the sum of these will as much exceed 50 shillings,

as the sum of the prices of the greatest and middle is below 25 shillings; what did each cost?

Let $12x =$ price of the first, then $5x =$ price of middle,
 $x =$ price of least, $43x - 50 = 25 - 17x \therefore x = 1\frac{1}{4} =$
 1s. 3d. $5x = 6s. 3d. 12x = 15s.$

13. A man and his wife usually drink a cask of beer in 12 days, but when the man is from home, it will last his wife 30 days: In how many days would the man alone be drinking it?

Let $x =$ No. required. $\begin{matrix} \text{days.} & \text{cask.} & \text{days.} & \text{cask.} \\ x : 1 & :: 12 : \frac{12}{x} = \text{part} \end{matrix}$
 drunk by the man in 12 days.

$30 : 1 :: 12 : \frac{12}{30} = \text{part drunk by the woman in}$
 12 days. $\therefore \frac{12}{x} + \frac{12}{30} = 1$, whence $x = 20$.

14. A footman who agreed with his master for £8. a year, and his livery, was turned away at the end of 7 months, and received £2. 3s. 4d. besides his livery: What was its value?

Let $x =$ value required, $£2. 3s. 4d. = \frac{13}{6}£ \therefore 12$
 months : $£8. + x :: 7$ months : $\frac{13}{6}£ + x$, whence $x =$
 £6.

15. Two ships laden with wine sailing through a pass, must pay toll according to the quantity each has on board; A has 250 hogheads, out of which were given one hoghead, and 36 shillings; B has 400 hogheads, and gives 2 hogheads, receiving back

20 shillings: What was the wine valued at per hogs-head?

Let x = value. A paid $x + 36$ shill. B $2x - 20$ shill. $250 : x + 36 :: 400 : 2x - 20 \therefore 100x = 19400, x = 194 = £9 \ 14.$

16. Divide 72 into three parts, so that half of the first part, one third of the second, and one fourth of the third, may be equal to each other.

Let $2x$ = first part, $3x$ = second, $4x$ = third.

Their sum $= 9x = 72 \therefore x = 8.$

17. A market woman bought a certain number of eggs at 2 a penny, and as many at 3 a penny, and sold them all at the rate of 5 for two-pence, and by so doing, lost 4d. What number had she?

Let x = No. of each sort, then $\frac{x}{2}$ = price of first sort,

$\frac{x}{3}$ = ditto of second sort, $5 : 2 :: 2x$ (= No. of eggs)

$: \frac{4x}{5}$ = price at 5 for two-pence.

$\therefore \frac{x}{2} + \frac{x}{3} - \frac{4x}{5} = 4$, whence $x = 120.$

18. To divide 90 into 4 parts, so that if the first be increased by 2, the second diminished by 2, the third multiplied by 2, and the fourth divided by 2, the sum, difference, product, and quotient, will be all equal to each other.

Let $x - 2$ = first part, then $x + 2$ = second, $\frac{x}{2}$ =

third, $2x$ = fourth, and their sum $4x + \frac{x}{2} = 90 \therefore x = 20.$

19. There is a number consisting of 2 figures, and it is equal to 4 times their sum, but if 18 be added to the number, the figures will change places, that is, the first will stand last, and the last first: What is the number?

Let $x =$ fig. in the tens place, $y =$ that in the units, then $10x + y =$ the No. $\therefore 10x + y = 4 \times (x + y) = 4x + 4y$, also $10x + y + 18 = 10y + x \therefore x = 2, y = 4, 10x + y = 24, 10y + x = 42.$

20. Two men set out from the same place and travel the same way, A goes 1 mile the first day, 2 the second, 3 the third, and so on; B sets out 5 days after A, and travels 12 miles a day; how long, and how far will A have travelled before B overtakes him?

Let $x =$ number of days A travelled, $x - 5 =$ the number of days B travelled $12x - 50 =$ miles travelled by B; $\left(\overline{x+1} \times \frac{x}{2}\right) \frac{x^2+x}{2} =$ miles travelled by A. $\therefore$

$$\frac{x^2 + x}{2} = 12x - 50, x = 8 \text{ days, } 12x - 50 = 36 \text{ miles.}$$

21. Bought cheeses for £10. now if 5 more could have been bought for the same money, the whole would have cost £1. a piece less: How many were there?

$$\text{Let } x = \text{Number, then } \frac{10}{x} = \frac{10}{x+5} + 1 \therefore x = 5.$$

22. Sixty thousand soldiers were placed in the form of a rectangle, whose length was to its breadth, as 3 to 2; how many men were there, and how many acres of ground did they occupy, allowing $2\frac{3}{4}$ yards between man and man?

PROBLEMS.

247

Let $3x$ = number of men in length, then $2x$ = ditto breadth $\therefore (3x \times 2x =) 6x^2 = 60000$, $x = 100$, $3x = 300$ = length, $2x = 200$ = breadth. Also, $(300 - 1) \times 2,75 = 822,25$ yards in length, $(200 - 1) \times 2,75 = 547,25$ yards in breadth $\therefore 822,25 \times 547,25 = 449976,3125$ square yards = $92,970312$ acres, = 92 A. 3 R. $35,24992$ Perches.

23. Find 2 numbers such that the square of their product added to their 4th powers, will make 34048 , and the sum of their squares added to their product, 304 .

Let x and y = the numbers, then $x^4 + x^2y^2 + y^4 = 34048$, and $x^2 + xy + y^2 = 304$, dividing the 1st by the 2d, we have $x^2 - xy + y^2 = 112$, and taking this from the second, leaves $2xy = 192 \therefore xy = 96$, add this to the 2d, and subtract it from the 3d, whence $x^2 + 2xy + y^2 = 400$ and $x^2 - 2xy + y^2 = 16 \therefore x + y = 20$. $x - y = 4 \therefore x = 12$, $y = 8$.

24. There is a number from the cube of which 3 times its square being subtracted, the remainder shall exceed 3 times the said number by unity: What is the number?

Let x = number, then $x^3 - 3x^2 = 3x + 1$, or $x^3 = 3x^2 + 3x + 1$, add x^3 to both sides, and extract the cube root, and $x\sqrt[3]{2} = x + 1$, whence $x = \frac{1}{\sqrt[3]{2} - 1}$.

25. What 2 numbers are those whose product is 35 , and the difference of their cubes 218 ?

Let x = less, then $\frac{35}{x}$ = greater . $218 \left(= \left(\frac{35}{x} \right)^3 - x^3 \right)$
 $= \frac{42875}{x^3} - x^3 \therefore x^6 + 218x^3 = 42875$, now comp. the sq.

$$\text{and } x^5 + 218x^3 + (109)^2 = 11881 = 54756 : \therefore x^4 + 109 = 234. \therefore x = \sqrt[3]{125} = 5.$$

26. There are 3 numbers in the Geometrical Progression, whose product is 216, and the sum of their cubes 1009. What are the numbers?

$$\text{Let } x, xy \text{ and } xy^2 \text{ be the numbers, then } x^3 y^3 = 216 \\ x^3 + x^3 y^3 + x^3 y^6 = 1009, \text{ from the 1st } y^3 = \frac{216}{x^3} \text{ write}$$

$$\text{this for } y^3 \text{ in the 2d, and } x^3 + 216 + \frac{46656}{x^3} = 1009.$$

$$\therefore x^6 - 793x^3 = -46656, \text{ comp. the sq. \&c.}$$

$$x^3 - 396.5 = \pm \sqrt{110556.25}.$$

$$x = \sqrt[3]{396.5 - \sqrt{110556.25}} = 4. y^3 \left(= \frac{216}{x^3} = \frac{216}{64} \right) =$$

$$\frac{27}{8}.$$

$$y = \frac{3}{2} = 1.5, \text{ whence the numbers are 4, 6 and 9.}$$

27. Of four numbers, the sum of the first three is 13, the sum of the two first and last 17, the sum of the first and two last 20, and the sum of the three last is 22: what are the numbers?

Let $x, y, z,$ and $w,$ represent the numbers, and $s =$ their sum.

Then $s - w = 13, s - z = 17, s - y = 20, s - x = 22,$ and their sum $(4s - x - y - z - w) = 4s - s = 72$

$\therefore s = 24,$ substitute this value of s in the four first eq. and $24 - w = 13 \therefore w = 11; 24 - z = 17 \therefore z = 7; 24 - y = 20 \therefore y = 4; 24 - x = 22 \therefore x = 2.$

GENERAL PROBLEMS.

A General Problem is one in which all the quantities known as well as unknown, are represented by letters; such a general representation is of the greatest use, for by this means the solution becomes universal, supplying a rule for every particular problem of the sort, that can be proposed.

The process by which the value of the unknown quantity is found, is called the **ANALYSIS**, or **ANALYTICAL INVESTIGATION** of the problem.

Having found the value of the unknown quantity in known terms—the substituting this value in this given equation instead of the unknown quantity, and proving that the quantity so assumed has the properties described in the problem, is called the **SYNTHESIS** or **SYNTHETICAL DEMONSTRATION**.

The translation of the value of the unknown quantity (in known terms) out of algebraic into common language, in which the relation of the quantities is simply declared, is deducing a **THEOREM**, but if the translation be made in the form of a *precept* it becomes a **CANON** or **RULE**.*

* A *Theorem* and *Canon* deduced from problem 1, may be as follow:

THEOREM. Half the difference of two numbers added to half their sum, is equal to the greater; and half the difference subtracted from half the sum is equal to the less.

CANON or RULE. Add the sum and difference of two numbers together, and divide this sum by 2, the quotient will be the greater.

Subtract the difference from the sum, and divide the remainder by 2, the quotient will be the less.

In this manner Theorems and Rules may be deduced from every general algebraic process; most of the intricate rules, in arithmetic and other branches of computation, have been found by similar methods. See *Ludlam's Rudiments*, p. 99.

1. What two numbers are those whose sum is s and difference d ?

Let x = the greater, y = the less.

Then $x+y=s$
And $x-y=d$ } by the problem.

Whence, by adding, $2x=s+d$; and $x=\frac{s+d}{2}$.

And by subtracting, $2y=s-d$; and $y=\frac{s-d}{2}$.

Synthetical Demonstration.

Because $\frac{s+d}{2}$ and $\frac{s-d}{2}$ are the quantities found by the analysis, substitute them in the given equations instead of x and y , therefore $x+y=s$ becomes $\frac{s+d}{2} + \frac{s-d}{2} = \frac{2s}{2} = s$, and $x-y=d$ becomes $\frac{s+d}{2} - \frac{s-d}{2} = \frac{2d}{2} = d$, where it is to be observed, that by adding and subtracting these quantities, the results (s and d) are the very same as the problem requires, which is the proof.

2. Divide a given number a , into two parts, called x and y , so that r times x added to s times y , may make some other given number b .

Here $x+y=a$, and $rx+sy=b$, from the former $y=a-x$, and $sy=sa-sx$, substitute this for sy in the latter, and $rx-sx+as=b$, whence $rx-sx=b-as$, and $x=\frac{b-as}{r-s}$, and $y(=a-x)=a-\frac{b-as}{r-s}=\frac{ar-as}{r-s}-\frac{b-as}{r-s}=\frac{ar-b}{r-s}$.

GENERAL PROBLEMS.

251

Example. Divide 32 into two parts, so that 3 times the greater added to 4 times the less will give 108.

Here $a=32$, $r=3$, $s=4$, $b=108$.

$$\text{And } x = \frac{b-as}{r-s} = \frac{108-4 \times 32}{3-4} = \frac{108-128}{-1} = \frac{-20}{-1} = 20.$$

$$\text{And } y = \frac{ar-b}{r-s} = \frac{32 \times 3-108}{3-4} = \frac{96-108}{-1} = \frac{-12}{-1} = 12.$$

SYNTHESIS. First, $\frac{b-as}{r-s} + \frac{ar-b}{r-s} = \frac{ar-as}{r-s} = \frac{(r-s).a}{r-s} = a$, then $r \times \frac{b-as}{r-s} = \frac{br-ars}{r-s}$ ($=r$ times x). and $s \times \frac{ar-b}{r-s} = \frac{ars-bs}{r-s}$ ($=s$ times y). whence, $\frac{br-ars}{r-s} + \frac{ars-bs}{r-s} = \frac{(r-s).b}{r-s} = b$.

3. To divide a given number a into two such parts, that one may be to the other as r to s .

Let x and y be the parts, then $x+y=a$, and $x:y::r:s$, wherefore $sx=ry$, and from the former $y=a-x$, whence, by substitution, $sx=ar-rx$, consequently $rx+sx=ar$, and $x=\frac{ar}{r+s}$, whence $y(=a-x)=a-\frac{ar}{r+s} = \frac{ar+as-ar}{r+s} = \frac{as}{r+s}$.

Example. Let 20 be divided into 2 parts, which are to each other as 3 to 2.

Here $a=20$, $r=3$, $s=2$, $x=\frac{ar}{r+s} = \frac{20 \times 3}{3+2} = \frac{60}{5} = 12$, and $y=\frac{as}{r+s} = \frac{20 \times 2}{3+2} = \frac{40}{5} = 8$.

SYNTHESIS. First, $\frac{ar}{r+s} + \frac{as}{r+s} = \frac{ar+as}{r+s} = a$.

Then, $\frac{ar}{r+s} : \frac{as}{r+s} :: ar : as :: r : s$.

4. Given the sum s , and the sum of the cubes b , of two numbers x and y , to find them.

Here $x+y=s$, and $x^3+y^3=b$, from the first $y=s-x$, whence $y^3=s^3-3s^2x+3sx^2-x^3$, substitute this in the second, and $s^3-3s^2x+3sx^2=b$, whence $-3s^2x+3sx^2=b-s^3$, or $3sx^2-3s^2x=b-s^3$, wherefore $x^2-sx=\frac{b-s^3}{3s}$, and completing the square $x^2-sx+\frac{s^2}{4}=\frac{b-s^3}{3s}+\frac{s^2}{4}$, substitute $\frac{R^2}{4}$ for the known side of this equation, and

it will become $x^2-sx+\frac{s^2}{4}=\frac{R^2}{4}$, whence $x-\frac{s}{2}=\pm\frac{R}{2}$, and $x=\frac{s+R}{2}$, also $y(=s-x)=s-\frac{s+R}{2}=\frac{s-R}{2}$.

But, $\frac{R^2}{4}=\frac{b-s^3}{3s}-\frac{s^2}{4}$, and $R^2=\frac{4b-s^3}{3s}$, whence, by making $R^2=\frac{4b-s^3}{3s}$, the numbers will be $\frac{s+R}{2}$.

Example. Let the sum of two numbers be 6, and the sum of their cubes 72.

Here $s=6$, $b=72$, $R^2=\frac{288-216}{18}=\frac{72}{18}=4$, and $R=2$, whence $x=\frac{s+R}{2}=\frac{6+2}{2}=4$, and 2, the numbers required.

*SYNTHESIS. First, $\frac{s+R}{2} + \frac{s-R}{2} = \frac{2s}{2} = s.$

$$\text{Then, } \left[\frac{s+R}{2} \right]^3 + \left[\frac{s-R}{2} \right]^3 = \frac{s^3 + 3s^2R + 3sR^2 + R^3}{8} + \frac{s^3 - 3s^2R + 3sR^2 - R^3}{8} = \frac{2s^3 + 6sR^2}{8} = \frac{s^3 + 3sR^2}{4}$$

restore the value of $3sR^2 (=4b-s^3)$ and $\frac{s^3 + 4b - s^3}{4} =$

$$\frac{4b}{4} = b.$$

†The method of registering the steps of an operation.

The register is a method of pointing out from whence any step is derived, and by what operation it is produced, by means of symbols placed opposite in the margin.

The mark $\odot$ denotes involution, ω evolution, $c\Box$ complete the square, tr. transpose; and in the following example the first column contains an explanation in words, the second the same in symbols, the third the number of each step, and the fourth the algebraic solution; also, in the register, the figures 1, 2, 3, &c. denote the first, second, third, &c.

* From these specimens the learner will sufficiently understand the method, so as to be able to demonstrate the following problems synthetically without farther assistance.

† This method was first introduced by Dr. Pell, and given by him in *Rhobius's Algebra*, printed at London in 1668. In long and intricate computations it will be found much more convenient than verbal references and explanations

steps; but a figure with a dash over the top signifies a number; thus, 3 denotes the 3d step, but $\overline{3}$ denotes the number 3.

EXAMPLE.

3. Given $\frac{x}{b} + \frac{y}{d} = m$ and $\sqrt{\frac{xy}{p}} = b$ to find x and y .

Explanation.	Regif. No	Operation.
Given in the problem.	$\left. \begin{array}{l} 1 \\ 2 \end{array} \right\}$	$\frac{x}{b} + \frac{y}{d} = m$ $\sqrt{\frac{xy}{p}} = b$
By transposing $\frac{y}{d}$ in eq. 1 . . .	$1 - \frac{y}{d}$	$\frac{x}{b} = m - \frac{y}{d}$
Multiplying eq. 3 into b . . .	$3 \times b$	$x = bm - \frac{by}{d}$
Involving eq. 2 to the square .	$2 \odot 2$	$\frac{xy}{p} = b^2$
Multiplying eq. 5 into p . . .	$5 \times p$	$xy = pb^2$
Dividing eq. 6 by y	$6 \div y$	$x = \frac{pb^2}{y}$
Equating the 4th and 7th eq. . .	$4 = 7$	$bm - \frac{by}{d} = \frac{pb^2}{y}$
Multiplying eq. 8 into y . . .	$8 \times y$	$bmy - \frac{by^2}{d} = pb^2$
Multiplying eq. 9 into d . . .	$9 \times d$	$dbmy - by^2 = dpb^2$
Dividing eq. 10 by b	$10 \div b$	$dmy - y^2 = dpb$
Transposing in eq. 11	11 tr.	$y^2 - dmy = -dpb$
Completing the sq. in eq. 12 . .	12 c □	$y^2 - dmy + \frac{d^2m^2}{4} = \frac{d^2m^2}{4} - dpb$
Extracting the sq. root in eq. 13	13 uuz	$y - \frac{dm}{2} = \pm \sqrt{\frac{d^2m^2}{4} - dpb}$
Transposing in eq. 14	14 tr.	$y = \frac{dm}{2} \pm \sqrt{\frac{d^2m^2}{4} - dpb}$

GENERAL PROBLEMS.

255

6. There are three numbers in geometrical proportion, their sum is $s(74)$, and the sum of their squares $b(1924)$, what are the numbers?

Let x, y, z be the respective numbers.

Then by {	1	$x+y+z=s$
the prob. {	2	$x^2+y^2+z^2=b$
	3	$x:y::y:z$
3 $\therefore$	4	$xz=y^2$
1—y	5	$x+z=s-y$
2—y ²	6	$x^2+z^2=b-y^2$
4×2	7	$2xz=2y^2$
6+7	8	$x^2+2xz+z^2=b+y^2$
5⊖2	9	$x^2+2xz+z^2=s^2-2sy+y^2$
8=9	10	$b+y^2=s^2-2sy+y^2$
10+	11	$2sy=s^2-b$
11÷2s	12	$y=\frac{s^2-b}{2s}=24$
4×4	13	$4xz=4y^2$
8—13	14	$x^2-2xz+z^2=b-3y^2$
14uu 2	15	$x-z=\sqrt{b-3y^2}$
5+15	16	$2x=s-y+\sqrt{b-3y^2}$
16÷2	17	$x=\frac{s-y+\sqrt{b-3y^2}}{2}=32$
5—x	18	$z=s-y-x=18$

7. A man playing at hazard, won at the first throw just as much money as he had; he won the second throw the square root of what he then had, and 5s. more; the third throw he won the square of all he then had; after which his whole sum was £112. 16s. how much had he at first?

*Let	1	$\frac{x^2}{2} = \text{his first sum.}$
1×2	2	$x^2 = \text{sum after 1st throw.}$
	3	$x+5 = \text{winnings at 2d throw.}$
$2+3$	4	$x^2+x+5 = \text{sum after 2d throw.}$
Substitute	5	$z = x^2+x+5$
$5 \odot 2$	6	$z^2 = \text{winnings at 3d throw.}$
$5+6$	7	$z^2+z = (2256 \text{ shill.}) = a \text{ by substit.}$
$7, \text{C} \square$	8	$z^2+z+\frac{1}{4} = a+\frac{1}{4} = \frac{R^2}{4} \text{ by subst.}$
$8 \text{ uu } 2$	9	$z+\frac{1}{2} = \pm \sqrt{\frac{R^2}{4}} = \pm \frac{R}{2}$
$9 - \frac{1}{2}$	10	$z = \frac{R}{2} - \frac{1}{2} = \frac{R-1}{2}$
$5 \dots 10$	11	$x^2+x+5 = \frac{R-1}{2}$
$11 - 5$	12	$x^2+x = \frac{R-1}{2} - 5 = \frac{R-11}{2}$
$12, \text{C} \square$	13	$x^2+x+\frac{1}{4} = \frac{R-11}{2} + \frac{1}{4} = \frac{S^2}{4} \text{ by subst.}$
$13 \text{ uu } 2$	14	$x+\frac{1}{2} = \pm \sqrt{\frac{S^2}{4}} = \frac{S}{2}$
$14 - \frac{1}{2}$	15	$x = \frac{S-1}{2} =$
But 13	16	$S = \sqrt{2R-21}$
8	17	$R = \sqrt{4a+1}$
By restitution	18	$x = \frac{S-1}{2} = \frac{\sqrt{2R-21}-1}{2} = \frac{\sqrt{2\sqrt{4a+1}-21}-1}{2}$
		$= \frac{\sqrt{2\sqrt{9025-21}-1}}{2} = \frac{\sqrt{169-1}}{2} =$
		$\frac{12}{2} = 6$
Whence	19	$\frac{x^2}{2} = \frac{36}{2} = 18 \text{ shillings.}$

* If the learner approves of the register, he may perfect himself in the use of it, in the resolution of the following problems, to which it may be applied.

8. Having the Principal, Rate per Cent per Annum, and time given to find the simple interest or amount of any sum, as also to find each of the others, (viz. principal, rate, and time) by having any three of the remaining quantities given.

Let p = principal or sum lent, r = ratio of the rate per cent. or the interest of 1£. for a year, t = time in years, or parts of a year, a = amount or principal and interest together, then 1£ : r :: p £ : pr = the interest of p pounds for a year, likewise 1 year : pr :: t years prt = the simple interest of p pounds for t years, at r per cent. per annum, and if p be added to this, the sum will be the amount, which is expressed in the following theorem.

THEOREM 1. $a = prt + p$.

From whence by transposition we get the following,

THEOREM 2. $p = \frac{a}{tr + 1},$

THEOREM 3. $r = \frac{a - p}{pt}.$

THEOREM 4. $t = \frac{a - p}{pr}.$

EXAMPLES.

1. What is the amount of 320£. for 3 years at 4 per cent per annum?

Here $p = 320$, $r = .04$ $t = 3$.

Whence by Theorem 1, $320 \times 3 \times .04 + 320 = 358.4$ £. = £358.8 the amount required.

2. What principal being put to interest at 5 per cent per annum, will amount to £1000 in 12 years?

Here $r = .05$ $a = 1000$ $t = 12$.

And Theorem 2. $\frac{1000}{12 \times .05 + 1} = \frac{1000}{1.6} = £625.$ the principal required.

3. At what rate per cent. will £210, amount to £399. in 20 years?

Here $p=210$, $a=399$, $t=20$.

Whence by Theor. 3, $\frac{399-210}{210 \times 20} = \frac{189}{4200} = .045$, or $4\frac{1}{2}$ per cent per annum.

4. In how many years will £210. amount to £399 at $4\frac{1}{2}$ per cent per annum?

Here $p=210$, $a=399$, $r=.045$.

And by Theorem 4, $\frac{399-210}{210 \times .045} = \frac{189}{9.45} = 20$ years.

9. To find the discount and present worth of any given sum of money payable some time hence, having the rate and time given.

The given sum may be considered as the amount of some principal (called here the present worth) at the given rate and time.

Let s = given sum, t and r as in the preceding Problem.

Then will $1+tr$ be the amount of 1£. hence

$$1+tr : 1 :: s : \frac{s}{1+tr} = \text{present worth of } s \text{ pounds.}$$

And

$$1+tr : tr :: s : \frac{str}{1+tr} = \text{discount.}$$

EXAMPLES.

1. What is the present worth of £405, 10. due four years hence at 5 per cent per annum?

Here $s=405, 10$, $t=4$, $r=.05$.

$$\text{And } \frac{s}{1+tr} = \frac{405,5}{1+4 \times ,05} = \frac{405,5}{1,2} = 337,9166, \text{ \&c.}$$

$\pounds = \pounds 337,18,4 = \text{present worth required.}$

2. What is the discount of $\pounds 150$ due 9 months hence, at 5 per cent per annum?

Here $s=150$ $t=(9 \text{ months})=.75$ $r=,05$.

$$\text{And } \frac{str}{1+tr} = \frac{150 \times ,75 \times ,05}{1+,75 \times ,05} = \frac{5,625}{1,0375} = 5,4216 \pounds$$

$= \pounds 5,8,5 = \text{the discount required.}$

10. To find the amount of any sum of money at Compound Interest.

Let p = principal, r = ratio of the rate per cent, $R = 1+r$ = the amount of $\pounds 1$ for 1 time, (as 1 year, &c.)
 a = the whole amount, t = time, L , logarithm.

Then as $\pounds 1 : R :: p : pR = 1^{\text{st}} \text{ years amount.}$

$\pounds 1 : R :: pR : pR^2 = 2^{\text{d}} \text{ years amount.}$

$\pounds 1 : R :: pR^2 : pR^3 = 3^{\text{d}} \text{ years amount.}$

$\pounds 1 : R :: pR^{n-1} : pR^n = n^{\text{th}} \text{ years amount.}$

From whence the following theorems are deduced, by which having any three of the above quantities given, the rest may be found.

THEOREM 1. $a = pR^t$.

THEOREM 2. $p = \frac{a}{R^t}$.

THEOREM 3. $R = \sqrt[t]{\frac{a}{p}}$.

*THEOREM 4. $t = \frac{L.a - L.p}{L.R}$.

* The example depending on this theorem may be omitted, if the learner does not understand logarithms.

EXAMPLES.

1. What is the amount of 500*£*. for 4 years, at 5 per cent per annum compound interest?

Here $p=500$ $t=4$ $R=1,05$, and

$$\text{Theor. 1. } a=500 \times 1,05^4 = 607,753125 = \text{£}607.15s. 0\frac{3}{4}d.$$

2. What sum at 4 per cent per annum, compound interest, would amount to 200*£*. in 3 years?

Here $a=200$, $R=1,04$, $t=3$, and

$$\text{Theor. 2. } p = \frac{200}{1,04^3} = \frac{200}{1,124864} = 177,7992 = \text{£}177.15s. 11\frac{3}{4}d.$$

3. At what rate per cent would *£*500 amount to *£*578. 15*s*. 3*d*. in 3 years, compound interest?

Here $p=500$ $a=578,8125$, $t=3$, and

$$\text{Theor. 3. } R = \sqrt[3]{\frac{578,8125}{500}} = \sqrt[3]{1,157625} = 1,05.$$

Whence $r=.05 = 5$ per cent per annum.

4. In what time will *£*225. amount to *£*260. 9*s*. 3*d*. at 5 per cent per annum compound interest?

Here $a=260,465625$, $p=225$, $R=1,05$, and

$$\text{Theor. 4. } t = \frac{L\ 260,465625 - L\ 225}{L\ 1,05} = \frac{2,41570 - 2,35218}{0,02119} = \frac{,06352}{,02119} = 3 \text{ years.}$$

11. To mix wine at a shillings a gallon with wine at b shillings a gallon, so that the mixture may contain c gallons at n shillings per gallon?

Let x = the number of gallons at a shillings per gallon, then $c - x$ = the number at b shillings per gallon, ax = price of the first sort, $b \times (c - x) = bc - bx$ = price of the other, cn = price of c gallon at n shillings per gallon, $\therefore ax + bc - bx = cn$ and $x = \frac{cn - bc}{a - b}$

Example. How much wine at 7 shillings and 12 shillings a gallon must be taken to make a mixture of 30 gallons, worth 10 shillings a gallon?

Here $a = 7$, $b = 12$, $c = 30$, $n = 10$, whence

$$x = \frac{30 \times 10 - 12 \times 30}{7 - 12} = \frac{-60}{-5} = 12 \text{ gallons at 7 shillings.}$$

And $c - x = 30 - 12 = 18$ gallons at 12 shillings.

12. What two numbers are those which are to one another as a to b , and the sum of whose squares is c ?

Let x = the less, then $a : b :: x : \frac{bx}{a}$ = the greater

therefore by the prob. $x^2 + \frac{b^2 x^2}{a^2} = c$, whence $x = \sqrt{\frac{a^2 c}{a^2 + b^2}} = a \sqrt{\frac{c}{a^2 + b^2}}$ = the less, from whence the greater may be found.

13. What two numbers are those whose difference is d , and the product of their cubes p ?

Let x = the less, then $x + d$ = the greater, and $x^3 \times x + d^3 = p$ by the prob. whence $x \times x + d = \sqrt[3]{p}$ on

$x^2 + dx = \sqrt[3]{p}$, this resolved gives $x = \sqrt{\frac{dd}{4}} + \sqrt[3]{p} -$

$\frac{d}{2}$ and $x + d = \sqrt{\frac{dd}{4}} + \sqrt[3]{p} + \frac{d}{2}$.

14. What two numbers are as m to n , and the sum of whose cubes is s ?

Let $x =$ the greater, then $m : n :: x : \frac{nx}{m}$ the less,

whence $x^3 + \frac{n^3 x^3}{m^3} = s$ by the prob. and $x = \sqrt[3]{\frac{s}{m^3 + n^3}}$

15. What two numbers are these, the square of the greater of which multiplied into the lesser produces a , and the square of the less multiplied into the greater b ?

Let $x =$ the greater, $y =$ the less, then $x^2 y = a$ and $xy^2 = b$, and dividing the first by the second $\frac{x}{y} = \frac{a}{b}$

and $= \frac{ay}{b}$. this value of x substituted in the second

eq. we have $\frac{ay^3}{b} = b$, whence $y = \sqrt[3]{\frac{b^2}{a}}$ and $x (= \frac{ay}{b})$

$= \frac{a}{b} \sqrt[3]{\frac{b^2}{a}}$

16. To divide the number a into two such parts that their product and the difference of their squares may be equal to each other.

Let $x =$ one part, then $a - x =$ the other, and $(a - x)x = (a - x)^2 - x^2$, that is $ax - x^2 = a^2 - 2ax$, which

resolved gives $x = \frac{3a}{2} \pm \sqrt{\frac{5a^2}{4}}$ and $a - x = -\frac{a}{2} \mp$

$\sqrt{\frac{5a^2}{4}}$ the number required.

17. Divide the number n into two parts so that the product of the squares of those parts may be p ?

Let x = one part, then $n - x$ = the other, and $x^2 \times (n - x)^2 = p$ by the prob. whence $x \times (n - x) = \sqrt{p}$, that is $nx - x^2 = \sqrt{p}$, from the resolution of which $x =$

$$\frac{n}{2} \pm \sqrt{\frac{nn}{4} - \sqrt{p}}.$$

18. To find two numbers whose sum, product, and the difference of their squares are all equal to each other.

Let x = the greater, y = the less, then by the prob. $x + y = xy$, and $x + y = x^2 - y^2$, the latter of these divided by $x + y$ gives $1 = x - y$, whence $x = 1 + y$ this value substituted for x in the first equation $1 + 2y = y + y^2$, whence $y = \frac{1}{2} + \sqrt{\frac{5}{4}}$ and $x (= 1 + y) =$

$$\frac{3}{2} + \sqrt{\frac{5}{4}}$$

19. Divide the number a into two such parts, that the sum of their square roots may be b .

Let x = the greater part, then $a - x$ = the less, and by the prob. $\sqrt{x} + \sqrt{a - x} = b$, and squaring both sides $x + 2\sqrt{ax - xx} + a - x = b^2$ whence $\sqrt{ax - xx} = \frac{b^2 - a}{2}$ again squaring both sides $ax - x^2 = \frac{b^2 - a}{2}$

whence $x = \frac{a}{2} + \frac{b}{2} \sqrt{2a - bb}$, and $a - x = \frac{a}{2} - \frac{b}{2} \sqrt{2a - bb}$.

20. Given the product p , and the sum of the fourth powers s of two numbers to find them.

Let x = the greater, y = the less, then $xy = p$ and $x^4 + y^4 = s$, add twice the square of the first equation to and subtract it from the second, and extract the square root of the sum and difference from, and we shall have $x^2 + y^2 = \sqrt{s + 2pp}$ and $x^2 - y^2 = \sqrt{s - 2pp}$, these two equations added together we get $x =$

$\sqrt{(\frac{1}{2}\sqrt{s + 2pp} + \sqrt{s - 2pp})}$, whence $y = \frac{p}{x}$ may be found.

21. A company of men put n pounds in the lottery, each contributed as many pounds as there were men, how many men were there?

Let x = the number of pounds each contributed, then as 1 man : x pounds : x men : x^2 pounds; hence $x^2 = n$ and $x = \sqrt{n}$.

22. To divide the number a into four such parts, that if the first be increased by b , the second decreased by c , the third multiplied into d , and the fourth divided by e , the result in each case will be the same.

Let x, y, z , and u be the parts, then $x + y + z + u = a$, and $x + b = y - c = dz = \frac{u}{e}$, whence $x = dz - b$

$y = dz + c$, $u = dex$, substitute these values for x, y and u , in the first equation, and $dz - b + dz + c + z + dex =$

a , whence $z = \frac{a + b - c}{de + 2d + 1}$ from which $x (= dz - b)$.

$y (= dz + c)$ and $u (= dex)$ may be found.

23. The sum of the squares $=a$, and the continual product $=b$ of four numbers in arithmetical progression being given to find the numbers.

Let $2x$ = the common difference, and $y-3x$ = the left extreme, then will $y-x$, $y+x$ and $y+3x$ be the other 3 terms, whence by the prob. $(y-3x)^2 + (y+x)^2 + (y+3x)^2 = a$.

$$\text{And } (y-3x) \cdot (y-x) \cdot (y+x) \cdot (y+3x) = b.$$

Which reduced gives $4y^2 + 20x^2 = a$.

$$\text{And } y^4 - 10y^2x^2 + 9x^4 = b.$$

From the former $y^2 = \frac{1}{4}a - 5x^2$, consequently $y^4 =$

$$\frac{1}{16}a^2 - \frac{5}{2}ax^2 + 25x^4 \text{ these values being substituted}$$

for y^2 and y^4 in the latter, we have $\frac{1}{16}a^2 - \frac{5}{2}$

$$ax^2 + 25x^4 - \frac{5}{2}ax^2 + 50x^4 + 9x^4 = b, \text{ from whence}$$

$$x^4 - \frac{5ax^2}{84} = \frac{b}{84} - \frac{a^2}{16 \times 84} \text{ this resolved gives } x =$$

$$\sqrt{\frac{5a \pm 2\sqrt{84b + a^2}}{168}} \text{ whence } y = \sqrt{\frac{1}{4}a - 5x^2} \text{ is also}$$

known.

24. The difference of the means $=a$, and the difference of the extremes $=b$ of four numbers in continued geometrical proportion, being given to find the numbers.

A a

Let $x = \text{sum of the means}$, then will $\frac{x+a}{2} = \text{the great-}$
er, and $\frac{x-a}{2} = \text{the less}$, whence $\frac{x+a}{2} : \frac{x-a}{2} :: \frac{x-a}{2} :$
 $\frac{\frac{x-a}{2}}{2x+2a}$ *the lesser extreme*, and $\frac{x-a}{2} : \frac{x+a}{2} :: \frac{x+a}{2} :$
 $\frac{\frac{x+a}{2}}{2x-2a}$ *the greater extreme*, whence by the problem,
 $\frac{\frac{x+a}{2}}{2x-2a} - \frac{\frac{x-a}{2}}{2x+2a} = b$, which resolved gives $x =$
 $a\sqrt{\frac{b+a}{b-3a}}$

25. The sum $= a$ and the sum of the squares $= b$; of three numbers in geometrical proportion being given to find them.

Let x, y , and z denote the numbers respectively.

Then will $x+y+z=a$.

$$x^2+y^2+z^2=b.$$

And likewise $xz=y^2$.

Transpose y in the first equation, and square both sides, from which square subtracting the second equation, we have $2xz-y^2=a^2-2ay+y^2-b$, but by the third $2xz=2y^2$, whence $y^2=a^2-2ay+y^2-b$, and $y = \frac{a^2-b}{2a}$, and by the second equation, $x^2+z^2=b-y^2$, from which subtracting $2xz=2y^2$ there arises $x^2-2xz+x^2+z^2=b-3y^2$, from which extracting the square root we have $x-z=\sqrt{b-3y^2}$, but from equation 1 we

have $x+z=a-y$, whence by adding and subtracting these two last equations, we have $2x=a-y+\sqrt{b-3y^2}$ and $2z=a-y-\sqrt{b-3y^2}$.

26. To find three numbers in arithmetical progression whereof the sum of the squares shall be s , and the square of the mean shall exceed the product of the extremes by m .

Let x = the mean, z = common diff. then $x-z$, x and $x+z$ will be the numbers, but $x^2 = (x-z).(x+z) + m$ by the prob. that is $x^2 = x^2 - z^2 + m$, whence $z = \sqrt{m}$.

Now $x - \sqrt{m}^2 = x^2 - 2x\sqrt{m} + m = \text{sq. of } 1\text{st.}$

$x^2 = x^2 \quad \cdot \quad \cdot \quad \cdot = \text{sq. of } 2\text{d.}$

$x + \sqrt{m}^2 = x^2 + 2x\sqrt{m} + m = \text{sq. of } 3\text{d.}$

And their sum $2x^2 + 2m = s$, whence $x = \sqrt{\frac{s-2m}{2}}$.

27. Given the sum s , and product p of any two numbers, to find the sum of their squares, cubes, bi-quadrates, &c.

Let x and y be the numbers, then $x+y=s$, $xy=p$.

But $(x+y)^2 = x^2 + 2xy + y^2 = s^2$ by involution.

And $x^2 + 2xy + y^2 - 2xy = s^2 - 2p$ by subtraction.

That is, $x^2 + y^2 = s^2 - 2p = \text{sum of the squares.}$

Again $(x^2 + y^2) \times (x+y) = (s^2 - 2p) \times s$ because $x+y=s$.

Or $x^3 + xy \times (x+y) + y^3 = s^3 - 2sp$, which by substituting sp for its equal $xy \times (x+y)$ becomes $x^3 + sp + y^3 = s^3 - 2sp \therefore x^3 + y^3 = s^3 - 3sp = \text{sum of the cubes.}$

Also $(x^3 + y^3) \times (x + y) = (s^3 - 3sp) \times s$, or $x^4 + xy \times (x^2 + y^2) + y^4 = s^4 - 3s^2p$, which by substituting $p \times (s^2 - 2p)$ for its equal, $xy \times (x^2 + y^2)$ becomes $x^4 + p \times (s^2 - 2p) + y^4 = s^4 - 3s^2p \therefore x^4 + y^4 = s^4 - 3s^2p - p \times (s^2 - 2p) = s^4 - 4s^2p + 2p^2 = \text{sum of the biquadrates, in like manner it may be shewn, that } x^5 + y^5 = s^5 - 5s^3p + 5sp^2 = \text{sum of the 5th powers, and } s^n - n s^{n-2}p + n \cdot \frac{n-3}{2} s^{n-4}p^2 - n \cdot \frac{n-4}{2} \cdot \frac{n-5}{3} s^{n-6}p^3 + n \cdot \frac{n-5}{2} \cdot \frac{n-6}{3} \cdot \frac{n-7}{4} s^{n-8}p^4 - \&c. = \text{sum of the } n\text{th powers.}$

28. Given the sum of two numbers, and the sum of their biquadrates to find the numbers.

Put $a = \text{half the sum of the numbers}$, $x = \text{half their difference}$, $s = \text{sum of their fourth powers}$, then will $a + x$ and $a - x$ be the numbers respectively; and $(a + x)^4 + (a - x)^4 = s$, that is, $2a^4 + 12a^2x^2 + 2x^4 = s$; whence $x^4 + 6a^2x^2 = \frac{1}{2}s - a^4$; and comp. the sq. $x^4 + 6a^2x^2 + 9a^4 = \frac{1}{2}s + 8a^4$, consequently $x = \sqrt{-3a^2 + \sqrt{(\frac{1}{2}s + 8a^4)}}$ from which the numbers will be found.

29. Given the sum, and the sum of the fifth powers of two numbers to find them.

Let $s = \text{the sum of the fifth powers}$, and the rest as in the preceding problem, then we shall have $2a^5 + 20a^3x^2 + 10ax^4 = s$, which resolved gives $x = \sqrt{\sqrt{(\frac{s}{10a} + \frac{4a^4}{5})} - a^2}$.

30. Let the number of cards in a pack $= p$, be divided into any number of heaps n , so that the spots on the bottom card added to the number of cards laid thereon may be q , and the number of cards re-

maining in the dealer's hand r , to find the sum of the spots on all the bottom cards.

Let x = the sum required, also $\overline{q+1} \cdot n$ = the number of cards and spots in all the heaps, from which subtracting the number of spots, there will remain the number of cards only, that is $\overline{q+1} \cdot n - x = p - r$; whence $x = \overline{q+1} \cdot n - p + r$.

31. If a oxen in b weeks eat c acres of grass; and d oxen in n weeks eat m acres of the same; how many oxen will eat r acres in s weeks, the grass growing uniformly?

Let x = number sought, 1 = the grass eaten by an ox in 1 week, w = the grass on an acre at first, z = the increase per week per acre (after the first 5 weeks), $p=4$ (= the excess of 9 weeks above 5), $r=14$ (= the excess of 19 weeks above 5); then rxz = the increase on r acres in t weeks, rw = the grass on r acres at first, $\therefore sx = rw + rtz$. (eq. 1). Likewise, mpz = the increase on m acres in p weeks, and mw = the grass thereon at first. Now if 1 ox : eats 1 grass in 1 week $\therefore d$ oxen : will eat dn grass in n weeks, $\therefore mw + mpz = dn$, (eq. 2). also the grass on c acres at first = what a oxen eat in b weeks; $\therefore cw = ab$, $w = \frac{ab}{c}$

(eq. 3); now to eq. 1, multiplied into mp , add eq. 2, multiplied into rt , and $mpsx + mrtw + mprtz = dnr + mprw + mprtz$, which (by transposition, &c. and writing $\frac{ab}{c}$ for w) becomes $cmpsx = cdnr + abmpr -$

$abmrt$, whence $x = \frac{cdnr + abmpr - abmrt}{cmps}$.

Example. Suppose $a=18$, $b=5$, $c=6$, $d=45$,
 $m=21$, $n=9$, $r=38$, $s=19$, $t=14$, $p=4$, then will $x=$
 $\frac{574560}{9576} = 60$ oxen.

32. Suppose a waterman can row m miles an hour with the tide, and n miles an hour against it; and that he rows p miles up a river and back again in r hours, the tide running uniformly the same way; required the velocity of the tide?

Let v = velocity required, x = the time he rowed with the tide, then $r-x$ = the time he rowed against it.

hours. hour.

* Then $x : p :: 1 : \frac{p}{x}$ = his velocity with the

tide, and $r-x : p :: 1 : \frac{p}{r-x}$ = his velocity

against tide: but the tide affists him = v in going with it, and retards him = v , as he goes against it, consequently the diff. of his vel. = $2v$ $\therefore \frac{p}{x} - \frac{p}{r-x}$

= $2v$, or $v = \frac{p}{2x} - \frac{p}{2r-2x}$; now because his vel. with

tide is to his vel. against it as m to n , $\therefore$ his time of rowing with is to rowing against tide as n to m ;

$\therefore x : r-x :: n : m$ or $x = \frac{nr}{m+n}$

substitute this for x in the eq. $v = \frac{p}{2x} - \frac{p}{2r-2x}$ and we have

* Because the velocity is directly as the space described in a given time.

$$v = p \div \frac{2nr}{m+n} - p \div \left(2r - \frac{2nr}{m+n} \right) = \frac{mp+np}{2nr} -$$

$$\frac{mp+np}{2mr} = (m^2-n^2) \times \frac{p}{2mnr}$$

33. Suppose two luminaries joined by a straight line, to find a point in the same which is equally enlightened by both ?

Let a = their distance apart, x = the dist. of the lesser from the point required, then $a-x$ = dist. of the greater.

Now if the bodies are equal, the effects they produce will be as * $\frac{1}{x^2} : \frac{1}{(a-x)^2}$.

Let the less be to the greater as m to n .

Then will their effects be respectively as $\frac{m}{x^2} : \frac{n}{(a-x)^2}$

$$\therefore \frac{m}{x^2} = \frac{n}{a^2-2ax+x^2} \text{ or } mx^2-nx^2-2amx=-a^2m$$

$$\therefore x^2 - \frac{2am}{m-n}x = -\frac{a^2m}{m-n} \text{ now comp. the sq. and}$$

* The effects of light from the same or equal luminaries are reciprocally as the squares of their distances from the respective points illuminated.

† For the effects are directly as the bodies and inversely as the squares of the distances.

If instead of $a-x$ we had put $a+x$ for the distance of the greater, the solution would have been more general, including the case in which the point is beyond the smallest luminary, whereas the above supposes the point situated some where between them.

$$x^2 - \frac{2am}{m-n}x + \frac{a^2m}{m-n} = \frac{a^2m}{m-n} - \frac{a^2m}{m-n}.$$

$$\text{whence } x = \frac{am}{m-n} \pm \sqrt{\left(\frac{am}{m-n}\right)^2 - \frac{a^2m}{m-n}}.$$

34. The velocities of two moving bodies A and B, tending in opposite directions to the same point, together with the distance of places and times from, and in which they begin to move being given to find the point where they meet.

Let x = the dist. of A from the point where they meet, d = dist. of A from B, t = time between their setting out; also, let A move through the space a in the time n , and B through the space b in the time m ; then

$a : n :: x : \frac{nx}{a}$ = the time A moves; and $b : m ::$

$d-x : \frac{(d-x) \cdot m}{b}$ = the time B moves, wherefore

$\frac{nx}{a} - \frac{(d-x) \cdot m}{b} = t$ by the prob. which reduced gives

$$x = \frac{bt + dm}{bn + am} \times a.$$

Example. Two persons being 59 miles asunder, set out to meet each other, A goes 7 miles in 2 hours; and B (who sets out an hour later than A) goes 8 miles in 3 hours—how many miles must each travel before they meet?

Here $d=59$, $a=7$, $n=2$, $t=1$, $b=8$, $m=3$; and

$$x = \frac{1 \times 8 + 3 \times 59}{2 \times 8 + 7 \times 3} \times 7 = 35.$$

35 Given the velocities of two moving bodies A and B, tending in the same direction to the same place; together with the distance of the places, and times from and in which they begin to move; to find the place where they will be together.

Let x = the dist. of A (the farthest body) from the place required, d = dist. from A to B, t = interval of time between their setting out. Let A move through the space a in the time r , and B through the space b in the time s , then $a : r :: x : \frac{rx}{a}$ = time A moves; $b : s$

$:: x - d : \frac{(x-d) \cdot s}{b}$ = time B moves.

CASE 1. If A sets out before B.

Then $\frac{rx}{a} - \frac{(x-d) \cdot s}{b} = t$, which reduced gives $x = \frac{bt - sd}{br - as} \cdot a$.

CASE 2. If B sets out before A.

Then $\frac{(x-d) \cdot s}{b} - \frac{rx}{a} = t$, whence $x = \frac{bt + sd}{as - br} \cdot a$.

Example in case 1. B travelling 5 miles in 2 hours, sets out 4 hours after A, who was 59 miles behind him, and goes 10 miles in 3 hours, how many miles must A go to overtake B.

Here $a = 10, b = 5, r = 3, s = 2, t = 4, d = 59$.

and $x = \frac{5 \times 4 - 2 \times 59}{5 \times 3 - 10 \times 2} \times 10 = 196$ miles.

Example in case 2. B travelling 5 miles in 2 hours, sets out 4 hours before A, who was 59 miles behind

him, and goes 10 miles in 3 hours; how many miles must A travel to overtake B?

Here the rest of the letters being as before.

$$x = \frac{5 \times 4 + 2 \times 59}{10 \times 2 - 5 \times 3} \times 10 = 276 \text{ miles.}$$

36. The weight w of any mixture, together with the specific gravity thereof s , and of each of the two things mixed a and b , being given, to find the quantity of each.

Let x = wt. of that simple, whose sp. gravity a is greatest.

Then $w - x$ = wt. of the other.

$$\text{But } \left. \begin{array}{l} \frac{x}{a} = \\ \frac{w-x}{b} = \\ \frac{w}{s} = \end{array} \right\} \begin{array}{l} \text{the magnitude of the} \\ \text{body whose weight is} \end{array} \left\{ \begin{array}{l} x \\ w-x \\ w \end{array} \right.$$

Whence $\frac{x}{a} + \frac{w-x}{b} = \frac{w}{s}$, which resolved, gives

$$x = \frac{(s-b) \cdot aw}{(a-b) \cdot s}.$$

Example. What quantity of gold is there in a mixture of gold and silver, whose weight is 85 oz. and specific gravity 15; the specific quantity of gold being 19, and of silver $10\frac{1}{2}$?

Here $w=85$, $s=15$, $a=19$, $b=10\frac{1}{2}$.

$$\text{and } x = \frac{(15 - 10\frac{1}{2}) \times 19 \times 85}{(19 - 10\frac{1}{2}) \times 15} = 57 \text{ oz. of gold.}$$

37. The forces of several agents being given to determine the time wherein they will jointly produce a given effect e .

Let x = time required, and suppose

$$\left. \begin{array}{l} A \\ B \\ C \end{array} \right\} \text{ can perform } e \left\{ \begin{array}{l} a \\ b \\ c \end{array} \right\} \text{ times in the time } \left. \begin{array}{l} m \\ n \\ r \end{array} \right\} \\ \text{\&c.}$$

Then

$$\left. \begin{array}{l} m : a :: x : \frac{ax}{m} \\ n : b :: x : \frac{bx}{n} \\ r : c :: x : \frac{cx}{r} \end{array} \right\} \begin{array}{l} \text{The effect produced} \\ \text{in the times } x \text{ by} \end{array} \left\{ \begin{array}{l} A \\ B \\ C \end{array} \right.$$

$$\therefore \frac{ax}{m} + \frac{bx}{n} + \frac{cx}{r} = e. \quad x = \frac{e}{\frac{a}{m} + \frac{b}{n} + \frac{c}{r}}$$

Example 1. A vessel has 3 cocks. A can fill it in 1 hour, B in 2, and C in 3; in what time will they all fill it together?

Here $a=1, b=1, c=1, m=1, n=2, r=3, e=1$.

$$x = \frac{1}{\frac{1}{1} + \frac{1}{2} + \frac{1}{3}} = \frac{6}{11} \text{ hour.}$$

Example 2. A can perform a piece of work once in 3 weeks, B thrice in 8 weeks, and C 5 times in 12 weeks; in what time will they jointly perform the same?

GENERAL PROBLEMS.

Here $a=1$, $b=3$, $c=5$, $m=3$, $n=8$, $r=12$, $e=1$

$$x = \frac{1}{\frac{1}{3} + \frac{3}{8} + \frac{5}{12}} = \frac{8}{9} \text{ week.}$$

38. If two agents A and B can jointly produce an effect in the time m , A and C in the time n , and B and C in the time r ; in what time will each alone produce the same effect?

Suppose A could produce it in the time x , B in the time y , and C in the time z , and let 1 denote the effect.

Then

$$\left. \begin{array}{l} x : 1 :: m : \frac{m}{x} \\ y : 1 :: m : \frac{m}{y} \\ x : 1 :: n : \frac{n}{x} \\ z : 1 :: n : \frac{n}{z} \\ y : 1 :: r : \frac{r}{y} \\ z : 1 :: r : \frac{r}{z} \end{array} \right\} \begin{array}{l} \text{The effect produced by} \\ \left\{ \begin{array}{l} \text{A in the time } m \\ \text{B} \quad \quad \quad m \\ \text{A} \quad \quad \quad n \\ \text{C} \quad \quad \quad n \\ \text{B} \quad \quad \quad r \\ \text{C} \quad \quad \quad r \end{array} \right. \end{array}$$

$$\text{Whence } \frac{m}{x} + \frac{m}{y} = 1, \text{ or } \frac{1}{x} + \frac{1}{y} = \frac{1}{m} \text{ (Eq. 1.)}$$

$$\frac{n}{x} + \frac{n}{z} = 1, \text{ or } \frac{1}{x} + \frac{1}{z} = \frac{1}{n} \text{ (Eq. 2)}$$

$$\frac{r}{y} + \frac{r}{z} = 1, \text{ or } \frac{1}{y} + \frac{1}{z} = \frac{1}{r} \text{ (Eq. 3)}$$

$$\text{Their sum } \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right) \times 2 = \frac{1}{m} + \frac{1}{n} + \frac{1}{r}.$$

$$\text{Or } \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{2m} + \frac{1}{2n} + \frac{1}{2r} \text{ (Eq. 4.)}$$

And by subtracting Eq. 1, 2, and 3, from Eq. 4, thence will arise

$$\frac{1}{z} = \frac{1}{2m} + \frac{1}{2n} + \frac{1}{2r} - \frac{1}{m} = \frac{1}{2n} + \frac{1}{2r} - \frac{1}{2m} \text{ (Eq. 5.)}$$

$$\frac{1}{y} = \frac{1}{2m} + \frac{1}{2n} + \frac{1}{2r} - \frac{1}{n} = \frac{1}{2m} + \frac{1}{2r} - \frac{1}{2n} \text{ (Eq. 6.)}$$

$$\frac{1}{x} = \frac{1}{2m} + \frac{1}{2n} + \frac{1}{2r} - \frac{1}{r} = \frac{1}{2m} + \frac{1}{2n} - \frac{1}{2r} \text{ (Eq. 7.)}$$

$$\text{From Eq. 5. } z = \frac{2mnr}{nr + mn - mr}$$

$$\text{Eq. 6 } y = \frac{2mnr}{nr + mn - mr}$$

$$\text{Eq. 7. } x = \frac{2mnr}{mr + nr - mn}$$

Example. A and B can drink a barrel of beer in 60 days, A and C in 50, and B and C in 40. How many days will each alone require?

B b

Here $m=60$, $n=50$, $r=40$, $x=\frac{240000}{2400+2000-3000}$

$171\frac{3}{7}$ days. $y=\frac{240000}{2000+3000-2400}=92\frac{4}{13}$ days.

$z=\frac{250000}{2400+3000-200}=70\frac{10}{17}$ days

39. The hour and minute hands of a clock are exactly together at noon; when are they next together?

Ans. at $5\frac{5}{11}$ minutes past 1.

40. In what time will any sum of money double itself at 5 per cent. per annum simple interest?

Ans. 20 years.

41. A person has two horses, and a saddle worth £50. which if put on the back of the first horse, will make his value double that of the second; but if put on the back of the second, it will make his value triple that of the first; what is the value of each?

Ans. One £30. the other £40.

42. Old John who had in credit liv'd,
(Tho' now reduc'd) a sum receiv'd;
This lucky hit's no sooner found,
Than clam'rous duns came swarming round;
To Landlord—Baker—many more,
John paid in all—pounds ninety-four.
Half what remain'd—a friend he lent—
On Joan and self one fifth he spent;
And when of all these sums bereft,
One tenth o'th' sum receiv'd he'd left.
Say learned youths—(I know you can)
What sum reliev'd the good old man.

Ans. £141.

GENERAL PROBLEMS.

279

43. Sold a quantity of tobacco for 19 shillings—part of it at a shilling a pound, and the rest at 15 pence; now the first part was to the latter as $\frac{3}{4}$ to $\frac{2}{3}$; how much was there of each sort?

Ans. 9lb. at 1s. and 8lb. at 15d.

44. Required three numbers in Geometrical Progression, such that the difference of the first and second may be 6, and of the second and third 15?

Ans. 4, 10, and 25.

45. Divide £16. between A and B, so that the cube of A's share may exceed the cube of B's by 386.

Ans. A's share £9. B's £7.

46. Given the sum of three numbers in Harmonical Proportion=26, and their continued product=576; what are the numbers?

Ans. 12, 8, and 6.

47. To divide a given number N into 4 parts, so that if another number n be added to the first part, deducted from the second, multiplied into the third, and the fourth divided by it, the sum, difference, product, and quotient, shall be equal to each other.

Ans. $\frac{Nn}{(n+1)^2} - n, \frac{Nn}{(n+1)^2} + n, \frac{N}{(n+1)^2}$ and

$\frac{Nn \times n}{(n+1)^2} = \text{the parts required.}$

48. A grocer sold 80lb. of mace, and 100lb. of cloves for £65; but he sold 60lb. more of cloves for £20. than he did of mace for £10. what price did he sell each at?

Ans. Mace 10s. per lb. Cloves 5s. per lb.

B b 2

49. A bought 7 sheep more than B for £15. now the square of B's sheep was equal to the number of shillings A paid for each; how many did each buy?

Ans. A bought 12, B 5.

50. A and B owe between them £174. A pays £8. a day, and B pays 1 pound the first day, 2 pounds the second, 3 the third, and so on; in how many days will they clear the debt, and what sum did each pay?

Ans. 12 days. A paid £96. B £78.

51. Suppose 480 men are to be placed in the form of a rectangle, whose length and breadth together is to contain 52; how many men must there be in each?

Ans. 40 in length, 12 in breadth.

52. A lady being asked her age replied thus—

My age if multiplied by three,—

Two sevenths of the product tripled be;

The square root of two ninths of this is four—

Now tell my age, or never see me more.

Ans. 28 years.

53. Three gamblers staked 112 guineas at play, but disagreeing, each seized as many as he could; A got a certain number; B as many as A and 16 more—and C got a sixth part of their sum; how many had each?

Ans. A 40, B 56, and C 16.

54. A merchant began the world with a certain sum, out of which to maintain himself the first year, he spent £100. he traded with the rest, and at the end of the year had increased it by one third; he did so the second and third years, at the end of which his property was double what it was at first; what sum did he begin with?

Ans. £1480.

55. There is a wall whose length is 5 times its height, and its height 6 times its thickness, and it contains 4860 cubic feet of brick-work; to find its dimensions.

Ans. Length 90 feet, height 18 feet, thickness 3 feet.

56. A general disposing his army in the form of a square, finds that he has 284 men too many—he then increases the side by one man, and wants 25 to fill up the square; how many men were there?

Ans. 24000.

57. Given $x+y=a$, $x+z=b$, and $y+z=c$ to find x , y , and z .

Ans. $x = \frac{a+b-c}{2}$, $y = \frac{a+c-b}{2}$, $z = \frac{b+c-a}{2}$

58. A man dying left his wife with child, and ordered in his will that if the child proved a daughter she should have $\frac{1}{3}$ of his property, and the mother $\frac{2}{3}$, but if a son, he should have $\frac{2}{3}$, and his mother $\frac{1}{3}$ thereof; now it happened that the woman was delivered of a son and daughter; how must the estate (which was £6300.) be divided so as to answer the father's intentions?

Ans. $\left\{ \begin{array}{ll} \text{The daughter's share,} & \text{£900.} \\ \text{The mother's} & \text{£1800.} \\ \text{The son's} & \text{£3600.} \end{array} \right.$

59. There are four numbers in Harmonic Proportion whose sum is 55, and if the second be taken from twice the first, the remainder is 2, but if half the fourth be added to twice the first, the sum will be 25. What are the numbers?

Ans. 5, 8, 12, and 30.

60. If money be lent, at three per cent,
To those who chuse to borrow,
In what time shall I be worth a pound
If I lend a crown to morrow?

Ans. 46,90036 years, allowing comp. interest.

61. In what time will the simple interest of £400.
become equal to the discount of £520. at 5 per cent.
per annum?

Ans. 6 years.

62. Find the side of a cube whose superficies is to its
solidity as 6 to 11.

Ans. 11.

63. Two persons having an equal claim to an annuity of £100. to continue 30 years, agree to share it in this manner, A is to enjoy the whole annuity for the first 10 years; B and his heirs is to have the entire reversion thereof for the remaining 20 years.

Required the rate of compound interest allowed in this contract, and the present worth of the annuity?

Ans. The rate of interest £4. 18s. 7d. per cent. per ann. The present worth of the annuity £1549. 12s. 6d.

F I N I S.



